BERNSTEIN–ZELEVINSKY DUALITY FOR EQUIVARIANT VECTOR BUNDLES ON DRINFELD SPACES

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Abstract. We study duality and finiteness of locally analytic representations arising from compactly supported coherent cohomology of Drinfeld spaces. We prove that Grothendieck–Serre duality for equivariant vector bundles realizes Bernstein–Zelevinsky duality of locally analytic representations. We follow the strategy of Fargues for ℓ-adic sheaves and obtain resolutions of locally analytic representations by compactly induced representations. The resolutions motivate us to introduce the notion of locally analytic representations *of compact presentation*, which generalizes the notion of finitely presented representations. These representations are related to prim objects on the classifying stack in the sense of 6-functor formalism and their extension groups to admissible locally analytic representations are finite-dimensional. As an application, we establish Serre duality for extension groups of locally analytic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ attached to non-trianguline de Rham Galois representations under the p-adic local Langlands correspondence.

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1. Introduction

Let p be a prime number and let L be a finite extension of $\mathbb{Q}_p$. Let $n \geq 2$, and let $\mathcal{H}$ be the set of all L-rational hyperplanes in the analytic projective space $\mathbb{P}_L^{n-1,\mathrm{an}}$ over L. The Drinfeld space [Dri74]

$$\Omega_{\mathrm{Dr}} := \mathbb{P}_L^{n-1,\mathrm{an}} \setminus \cup_{H\in\mathcal{H}} H \tag{1.1}$$

and its étale covers, the Drinfeld towers, have played a crucial role in the Langlands program. These spaces admit natural actions of the p-adic Lie group $\mathrm{GL}_n(L)$ or $\mathrm{PGL}_n(L)$. Their cohomology realizes the local Langlands correspondence and the Jacquet–Langlands correspondence in both

the ℓ-adic case when $\ell \neq p$, e.g. [HT01, FGL08, Dat07], and in the p-adic case when $L = \mathbb{Q}_p$ and $n = 2$ [DLB17, CDN20].

In this paper, we consider compactly supported coherent cohomology of equivariant vector bundles on Ω_{Dr}. The study of coherent cohomology for Ω_{Dr} was initiated by Morita [MM82] and later by Schneider–Teitelbaum [ST02b], which motivated the development of their theory of locally analytic representations of p-adic Lie groups [ST02a, ST03]. Let $G = \mathrm{PGL}_n(L)$ and let $\mathcal{F}$ be a G-equivariant vector bundle on Ω_{Dr}. Since the rigid analytic space Ω_{Dr} is (quasi-)Stein [SS91, §1, Prop. 4], the compactly supported coherent cohomology $R\Gamma_c(\Omega_{\mathrm{Dr}}, \mathcal{F})$ concentrates in degree $n-1 = \dim \Omega_{\mathrm{Dr}}$ by Kiehl's Theorem B [Kie67b] and Serre duality for Stein spaces [Chi90, vdP92]. The cohomology group

$$H_c^{n-1}(\Omega_{\mathrm{Dr}}, \mathcal{F}) \simeq H^0(\Omega_{\mathrm{Dr}}, \mathcal{F}^\vee \otimes \omega)^\vee, \tag{1.2}$$

where ω denotes the canonical line bundle of Ω_{Dr}, is a locally analytic representation of the p-adic Lie group G. In the case $n = 2$, such representations are related to 2-dimensional de Rham p-adic Galois representations by the p-adic local Langlands correspondence, according to the Breuil–Strauch conjecture proved by Dospinescu–Le Bras [DLB17]. We refer to § 3.4 for more examples of equivariant vector bundles on Drinfeld spaces and to [Sch11, BQ26] for some work on the cohomology groups.

Our first result is the determination of the Bernstein–Zelevinsky dual of the G-representation (1.2). Let E be a finite extension of L. For V a locally analytic representation of G over E, its Bernstein–Zelevinsky dual is given by [SW25, Def. 5.4]

$$\mathbb{D}_{\mathrm{BZ}}(V) := R\mathrm{Hom}_G(V, \mathcal{C}_c^{la}(G, E)) \tag{1.3}$$

where $\mathcal{C}_c^{la}(G, E)$ denotes the space of compactly supported locally analytic functions on G, which is a G-bimodule via left and right G-translations. The definition extends the classical Bernstein–Zelevinsky duality (also called cohomological duality) for smooth representations of G in [Ber92, IV.5] and [Far] in which "locally analytic" is replaced by "smooth". Note that we work in the solid setting of [RJRC22, RJRC23] for the derived category of locally analytic representations.

Theorem 1.1 (Corollary 3.19)**.** *Let $G = \mathrm{PGL}_n(L)$ and let $\mathcal{F}$ be a G-equivariant vector bundle on Ω_{Dr}. Write $\mathbb{D}_{\mathrm{GS}}(\mathcal{F}) = \mathcal{F}^\vee \otimes \omega[n-1]$ for the Grothendieck–Serre dual of the vector bundle. Then*

$$\mathbb{D}_{\mathrm{BZ}}(R\Gamma_c(\Omega_{\mathrm{Dr}}, \mathcal{F})) \simeq R\Gamma_c(\Omega_{\mathrm{Dr}}, \mathbb{D}_{\mathrm{GS}}(\mathcal{F}))$$

as locally analytic representations of G.

In particular, $\mathbb{D}_{\mathrm{BZ}}(H_c^{n-1}(\Omega_{\mathrm{Dr}}, \mathcal{F})) \simeq H_c^{n-1}(\Omega_{\mathrm{Dr}}, \mathcal{F}^\vee \otimes \omega)[1-n]$ and $\mathbb{D}_{\mathrm{BZ}}(\mathbb{D}_{\mathrm{BZ}}(H_c^{n-1}(\Omega_{\mathrm{Dr}}, \mathcal{F}))) \simeq H_c^{n-1}(\Omega_{\mathrm{Dr}}, \mathcal{F})$. The above result is an analogue of the classical result for smooth G-representations and ℓ-adic local systems on Ω_{Dr} by Fargues [Far] (see [Mie14, Ham24] for generalizations). There, compactly supported cohomology of the Drinfeld tower intertwines the Bernstein–Zelevinsky duality and the Poincaré duality.

The proof of Theorem 1.1 is an adaptation of the argument of Fargues in [Far]. There exists an affinoid open subspace $U \subset \Omega_{\mathrm{Dr}}$ which is stabilized by an open compact subgroup of G such that the family of open subspaces $\{g(U)\}_{g \in G}$ forms a locally finite open covering of Ω_{Dr}. Then $R\Gamma_c(\Omega_{\mathrm{Dr}}, \mathcal{F})$ can be computed using two different Čech complexes with respect to the affinoid covering (corresponding respectively to !-descent and $*$-descent). The terms of the Čech complexes are finite sums of G-representations of the form $\text{c-Ind}_K^G H^0(V, \mathcal{F})$ or $\text{c-Ind}_K^G H_c^{n-1}(V, \mathcal{F})$ for some open compact subgroup $K \subset G$ and affinoid open $V \subset \Omega_{\mathrm{Dr}}$, where c-Ind_K^G denotes the compact induction. The key property of the Bernstein–Zelevinsky dual is [SW25, Thm. 5.3]:

$$\mathbb{D}_{\mathrm{BZ}}(\text{c-Ind}_K^G W) \simeq \text{c-Ind}_K^G W^\vee \tag{1.4}$$

when W is a Smith space so that its E-linear dual $W^\vee$ is a Banach space. Using (1.4) and Serre duality, one has $\mathbb{D}_{\mathrm{BZ}}(\text{c-Ind}_K^G H_c^{n-1}(V, \mathcal{F})) \simeq \text{c-Ind}_K^G H_c^{n-1}(V, \mathcal{F})^\vee \simeq \text{c-Ind}_K^G H^0(V, \mathcal{F}^\vee \otimes \omega)$. The result then follows from a comparison of the Čech complex for $R\Gamma_c(\Omega_{\mathrm{Dr}}, \mathbb{D}_{\mathrm{GS}}(\mathcal{F}))$ and the $\mathbb{D}_{\mathrm{BZ}}$-dual of that for $R\Gamma_c(\Omega_{\mathrm{Dr}}, \mathcal{F})$.

Remark 1.2. The arguments for Theorem 1.1 work for more general spaces (Theorem 3.16). However, to apply (1.4), we are forced to take U affinoid (instead of Stein). Consequently, we have to work within the framework of the analytic geometry of Clausen–Scholze [Sch19, CS22] and Rodríguez Camargo [RC24, RC26c] to use Serre duality for affinoid spaces. Along the way, we

prove results on compactly supported cohomology of smooth affinoid spaces (e.g. Proposition 2.28) which may be of independent interest.

The bonus of the proof of Theorem 1.1 is infinitely many resolutions of the G-representation $H_c^{n-1}(\Omega_{\mathrm{Dr}}, \mathcal{F})$ by representations of the form $\text{c-Ind}_K^G W$ where W is a Smith space representation of K. These W are of the form $H_c^{n-1}(V, \mathcal{F})$ where $V \subset \Omega_{\mathrm{Dr}}$ is affinoid. The resolutions motivate us to introduce the notion of locally analytic representations *of compact presentation*, which generalizes the notion of finitely presented (smooth) representations. We refer to Definition 4.1 for the precise definition. Roughly speaking, a locally analytic representation of compact presentation is a locally analytic representation that admits a finite resolution by representations of the form $\text{c-Ind}_K^G W$ where W is a Smith space representation of K, and one has certain freedom to change W to obtain other resolutions, corresponding to shrinking or enlarging the affinoid $V \subset \Omega_{\mathrm{Dr}}$ in the Drinfeld space cases. By the work of Kohlhaase–Schraen [KS12], extended in [SW25], locally analytic principal series representations are of compact presentation.

Proposition 1.3 (Proposition 4.7)**.** *Let $\mathcal{F}$ be a G-equivariant vector bundle on Ω_{Dr}. Then $H_c^{n-1}(\Omega_{\mathrm{Dr}}, \mathcal{F})$ is a locally analytic representation of G of compact presentation in the sense of Definition 4.1.*

The above result is an analogue of other finiteness results of cohomology groups of Drinfeld spaces, e.g. with mod p coefficients [CDN23, Thm. 0.5]. The following theorem is one of the main reasons that we introduce the notion of locally analytic representations of compact presentation.

Theorem 1.4 (Theorem 4.15)**.** *Let E be a finite extension of $\mathbb{Q}_p$ and let G be a p-adic Lie group. Let V be a locally analytic E-representation of G of compact presentation in the sense of Definition 4.1, and let Π be an admissible locally analytic representation of G in the sense of Schneider–Teitelbaum* [ST03]*. Then*

$$\dim_E \mathrm{Ext}_G^i(V, \Pi) < \infty$$

for every $i \geq 0$ and $\mathrm{Ext}_G^i(V, \Pi) = 0$ for all but finitely many $i \geq 0$.

Again, the above result generalizes a similar statement for finitely presented and admissible smooth representations. However, the proof for locally analytic representations is much more involved, relying on the theory of compact operators between Banach spaces. More precisely, we use a p-adic Schwartz lemma (Proposition 2.33) as in the proof of Cartan–Serre–Kiehl type finiteness theorem [Kie67a].

Remark 1.5. To the best of the authors' knowledge, finiteness results for extension groups of locally analytic representations as Theorem 1.4 were previously only available for particular representations before, e.g. [Din22] and [BQ26, Thm. 1.3.3].

The proof of Theorem 1.4 may remind experts of Emerton's finiteness (admissibility) result on the Jacquet–Emerton module [Eme06] and its derived version [Lee25] for admissible representations. However, even for V a locally analytic principal series representation, the finiteness of higher extension groups in Theorem 1.4 seems to be new, as the second adjointness (as in [Bre15, Prop. 4.2]) is not available for the derived Jacquet–Emerton module yet.

Remark 1.6. The classical Bernstein–Zelevinsky duality for finitely presented smooth representations can be studied from a more categorical point of view, for example from the 6-functor formalism in [HM24]. The notions of the Bernstein–Zelevinsky duality and compact presentation are related to "prim dual" and "prim objects" [HM24, § 4.4] in the derived category of solid locally analytic representations, which should play an important role in the categorical p-adic local Langlands correspondence, see [EGH25, § 6.2.1]. Some results of this paper are motivated by this perspective, and have analogues in the categorical setting [HM24, Lem. 4.4.9, Lem. 4.4.19]. This work serves as an effort to understand prim locally analytic representations in a representation-theoretic way:

Theorem 1.7 (Theorem 4.10)**.** *Suppose that V is a locally analytic representation of a p-adic Lie group.*

1. *The representation V is prim if and only if V is compact and its underlying solid $\mathbb{Q}_p$-vector space is nuclear. In this case, $\mathbb{D}_{\mathrm{BZ}}(V)$ coincides with its prim dual.*
2. *If V is of compact presentation (Definition 4.1), then V is prim.*

In particular, by Proposition 1.3, compactly supported coherent cohomology of Drinfeld spaces gives rise to prim representations. This result was previously obtained by Rodríguez Camargo [RC26b, Rem. 6.2.56] in a different way.

The last part of the paper focuses on the case $n = 2$ and $L = \mathbb{Q}_p$. Let E be a finite extension of $\mathbb{Q}_p$ and let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to \mathrm{GL}_2(E)$ be a de Rham Galois representation of Hodge–Tate weights $0, 1$. We assume that the Weil–Deligne representation $\mathrm{WD}(\rho)$ attached to ρ is absolutely irreducible and $\det(\rho)$ is the p-adic cyclotomic character ϵ. By [Col16, DLB17], the locally analytic representation $\pi(\rho)$ attached to ρ sits in a short exact sequence

$$0 \to \pi_{\mathrm{sm}} \to \pi(\rho) \to \pi_c \to 0$$

where π_{sm} is the smooth representation of $G = \mathrm{PGL}_2(\mathbb{Q}_p)$ attached to $\mathrm{WD}(\rho)$ by the classical local Langlands correspondence and $\pi_c \simeq H^1_c(\Omega_{\mathrm{Dr}}, \mathcal{F})$ for some G-equivariant vector bundle $\mathcal{F}$ on Ω_{Dr}. Thus we may apply Theorem 1.1 and Theorem 1.4 to π_c to calculate the Bernstein–Zelevinsky dual of $\pi(\rho)$ and obtain the following result on extension groups.

Theorem 1.8. *Let $\pi(\rho)$ be the locally analytic representation of $\mathrm{PGL}_2(\mathbb{Q}_p)$ attached to ρ. Then one has $\mathbb{D}_{\mathrm{BZ}}(\pi(\rho)) \simeq \pi(\rho)[-1]$. Moreover, for $0 \leq i \leq 3$, there is an isomorphism of finite-dimensional E-vector spaces*

$$\mathrm{Ext}^i_G(\pi(\rho), \pi(\rho)) \simeq \mathrm{Ext}^{3-i}_G(\pi(\rho), \pi(\rho))^\vee.$$

The duality of Ext groups is established using the Bernstein–Zelevinsky duality together with another duality $\mathbb{D}_{\mathrm{ST}}(-) \simeq (-)^{\vee, Rla}$, the Schneider–Teitelbaum duality for admissible locally analytic representations in [ST05] and [RJRC23, § 4.2]. The last duality is a generalization of the smooth duality (contragredient) for smooth representations. Similar to the smooth case (see e.g. [BK90], [GYD18, § 0.1], [Zhu25, Rem. 7.54] and [Han26]), the shift of the composition of the two duality functors $\mathcal{S}(-) := \mathbb{D}_{\mathrm{ST}}(\mathbb{D}_{\mathrm{BZ}}(-))[\dim G]$ behaves like a Serre functor (Corollary 5.24): we have

$$R\mathrm{Hom}_G(V_1, V_2)^\vee \simeq R\mathrm{Hom}_G(V_2, \mathcal{S}(V_1)) \tag{1.5}$$

for suitable locally analytic representations V_1 and V_2. The isomorphism (1.5) is a formal consequence of various properties of $\mathbb{D}_{\mathrm{BZ}}$ and $\mathbb{D}_{\mathrm{ST}}$. The representations in Theorem 1.8 are simultaneously of compact presentation and admissible, so that (1.5) can apply.

Remark 1.9. In contrast to compact presentation, the admissibility of coherent cohomology of the Drinfeld spaces is not formal and is not known in general. See different proofs for the admissibility of π_c in [DLB17, Pan26, AW23]. In § 5.2, we also determine the Schneider–Teitelbaum dual of $\pi(\rho), \pi_c$, using Colmez's construction of these representations from (φ, Γ)-modules over the Robba ring [Col16] together with Ding's study of their translations [Din23].

Remark 1.10. In a future work, we will use results in the paper to calculate all extension groups appearing in Theorem 1.8, and then determine the derived category generated by the relevant locally analytic representations, in the spirit of Paškūnas' work [Paš13]. The duality in Theorem 1.8 will correspond to the duality of Ext groups for modules over the Galois deformation rings.

Convention and Notation. Rigid analytic spaces are viewed as adic spaces. All adic spaces are assumed to be sheafy. For an adic space X, we write $|X|$ for the underlying topological space.

If A is an affinoid algebra, write A° for the subring of power bounded elements and $A^{\circ\circ} \subset A^\circ$ for the ideal of topologically nilpotent elements.

For a partially proper rigid analytic variety, we assume it is separated and countable at infinity. The last condition means that the space X can be written as a countable increasing union $X = \cup_n X_n$ such that each X_n is a quasi-compact open subspace of X and the inclusion $X_n \subseteq X_{n+1}$ factors through the universal compactification of X_n [Hub96, Def. 5.1.1].

A strict open immersion of rigid analytic spaces $j : U \hookrightarrow X$ is an open immersion such that j factors through the universal compactification of U.

We use the language of condensed mathematics [Sch19]. Throughout the paper, we fix an uncountable solid cutoff cardinal κ [Man22, Def. 2.9.11] and ignore the related set-theoretical issue. We denote the site of extremally disconnected sets by ExtDisc. For an (animated) analytic ring A, we write $D(A)$ for the stable ∞-category of A-modules and $\underline{A}$ for the underlying condensed

ring. If A is static ($\underline{A}$ sits in degree 0), then we write $\mathrm{Mod}(A)$ for the abelian category of A-modules. For a complete Huber pair (A, A^+), we use $(A, A^+)_\square$ to denote the associated analytic ring in [And21, § 3].

Let E be a finite extension of $\mathbb{Q}_p$, which will be our coefficient field, and let $\mathcal{O}_E$ be its ring of integers. We write $E_\square$ for the analytic ring $(E, \mathcal{O}_E)_\square$. We write $(R)\underline{\mathrm{Hom}}_E(-,-)$ for the internal hom in $\mathrm{Mod}(E_\square)$ (in $D(E_\square)$) and $\otimes^{(L)}_{E_\square}$ for the (derived) tensor product over $E_\square$. If $\mathcal{D}$ is a solid E-algebra, we write $\mathrm{Mod}_\square(\mathcal{D})$ (resp. $D_\square(\mathcal{D})$) for the category of (left) $\mathcal{D}$-module objects in $\mathrm{Mod}(E_\square)$ (resp. $D(E_\square)$) and $\otimes^{(L)\square}_{\mathcal{D}}$ for the relative tensor products.

Let Δ denote the simplex category, with objects $[n] = \{0, \dots, n\}$ and morphisms the order-preserving maps. If $Y \to X$ is a morphism of spaces, we write $(Y^{n/X})_{[n]\in\Delta^{\mathrm{op}}}$ or $(Y^{\bullet/X})_{\bullet\in\Delta^{\mathrm{op}}}$ for its Čech nerve, the simplicial object whose nth term $Y^{n/X}$ is the $(n+1)$-fold fibre product $Y \times_X \cdots \times_X Y$.

We freely use the results on solid functional analysis in [Bos23a, Appendix A], [RJRC22, § 3] and [CGN26, § 2.2]. Throughout, Banach spaces, Fréchet spaces and spaces of compact type (over $\mathbb{Q}_p$) denote their solid version, which are equivalent to the corresponding locally convex spaces by the results in *loc. cit.*. Especially, strict exact sequences of locally convex spaces of these types correspond to exact sequences of solid E-spaces. Recall Smith space over E are linear duals (via $\underline{\mathrm{Hom}}_E(-, E)$) of Banach spaces and they are compact project generators of $\mathrm{Mod}(E_\square)$ [RJRC22, Lem. 3.8].

If H is a p-adic Lie group, and E a finite extension of $\mathbb{Q}_p$, we write $\mathcal{C}^{la}(H, E)$ for the space of locally analytic functions on H. If H is compact, write $\mathcal{D}^{la}(H, E) \simeq \mathcal{C}^{la}(H, E)^\vee$ for locally analytic distribution algebra. In general take $H_0 \subset H$ an open compact subgroup and set $\mathcal{D}^{la}(H, E) = E[H] \otimes_{E[H_0]} \mathcal{D}^{la}(H_0, E)$. We view $\mathcal{D}^{la}(H, E)$ as a solid E-algebra. Let $\mathrm{Rep}^{la}_{E_\square}(H)^\heartsuit$ be the abelian category of locally analytic representations of G over solid E-spaces and $\mathrm{Rep}^{la}_{E_\square}(H)$ be its derived category, which is a full subcategory of $D_\square(\mathcal{D}^{la}(G, E))$ [RJRC23, Def. 3.3.1, Prop. 3.3.6]. The embedding $\mathrm{Rep}^{la}_{E_\square}(H) \hookrightarrow D_\square(\mathcal{D}^{la}(G, E))$ admits a right adjoint $(-)^{Rla}$, taking the derived locally analytic vectors [RJRC23, Cor. 3.3.7]. Admissible locally analytic H-representations in the sense of [ST03] are considered as solid locally analytic representations by [RJRC22, Prop. 4.42] and [RJRC23, Prop. 4.2.7].

If V is a left H-module for a group, then we also view V as a right H-module via the involution $\iota : h \mapsto h^{-1}$.

Overview. We first give a review of the basics on the analytic geometry and 6-functor formalism in § 2, then we use the machinery to prove the main duality theorem (Theorem 3.16) in § 3. In § 4, we define and study locally analytic representations of compact presentation. In § 5, we study supercuspidal locally analytic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. In Appendices A and B, we give a review on the Schneider–Teitelbaum duality and translation functors.

Declaration of AI use. We began to use AI in the later stages of the project, for literature search, proof exploration, TEX assistance, and proofreading. The main mathematical contribution of AI is an argument leading to the formulation and the proof of Lemma 4.14, a key homological lemma used in the proof of Theorem 4.15. The argument of AI is more direct than our original strategy; see Remark 4.17. AI also contributed to the proof of Proposition A.4, found the reference for Proposition 5.17, and suggested further corrections and improvements.

Acknowledgements. The third author would like to thank Matthias Strauch for discussions and the collaboration on [SW25] that led to this project. He would also like to thank Juan Esteban Rodríguez Camargo for very helpful discussions. The second author thanks the Tianyuan Mathematics Research Center for the opportunity to present some of the results of this paper in July 2026. The first author would like to acknowledge Simons Foundation and Institute for Advanced Study for their support, and thank Peking University for hospitality during his visit.

Part of the paper was written when the first author was in his Ph.D program at Sorbonne Université. This program has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 945332 and partially funded by the project "Group schemes, root systems, and related representations" funded by the European Union - NextGenerationEU through Romania's National Recovery and Resilience Plan

(PNRR) call no. PNRR-III-C9-2023- I8, Project CF159/31.07.2023. The second author was partially supported by the National Natural Science Foundation of China under Grant Nos. 12231001 and 12321001. This work emerged while the third author was a postdoc at the University of Münster, funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 427320536 – SFB 1442, as well as under Germany's Excellence Strategy EXC 2044 390685587, Mathematics Münster: Dynamics–Geometry–Structure.

2. Analytic geometry and Serre duality

In this section, we give a brief review of the rigid analytic geometry and the six functor formalism. As we will use the Serre duality for affinoid spaces, we will work in the framework of the analytic geometry developed by Clausen–Scholze in [CS22].

2.1. **Analytic adic space as categorified locale.** In the next two subsections we recall the Serre duality on smooth rigid analytic varieties proved in [RC24] and [RC26c].

For a presentably symmetric monoidal stable ∞-category C (see [RC24, § 1] for the notion), let $\mathcal{S}(C)^{\mathrm{op}}$ be the poset of isomorphic classes of idempotent algebras in C such that $A \leq A'$ if there is an arrow $A' \to A$ [CS22, Cons. 5.2]. By [CS22, Prop. 5.3], the poset $\mathcal{S}(C) := (\mathcal{S}(C)^{\mathrm{op}})^{\mathrm{op}}$ is a frame (i.e. a category that is similar to the category of open subsets of a topology space) and can be thought as open complements of $\mathcal{S}(C)^{\mathrm{op}}$. For $Z \in \mathcal{S}(C)^{\mathrm{op}}$ corresponding to an idempotent algebra A, we denote $C(Z) := \mathrm{Mod}_A(C) = \{X \in C \mid X \simeq X \otimes A\} \subset C$, the complement open of Z as $Z^c \in \mathcal{S}(C)$ and $C(Z^c) := C/C(Z)$, the Verdier quotient. We have the following localization functors:

Definition 2.1 ([RC24, Def. 2.2.2])**.** Let $U = Z^c \in \mathcal{S}(C)$ with associated idempotent algebra A, we have the following functors:

(1) Upper star functors: $i_Z^* : C \to C(Z) : M \mapsto M \otimes A$, and $j_U^* : C \to C(U)$ the natural projection.
(2) Lower star functors: $i_{Z*} : C(Z) \to C$ the forgetful functor, and $j_{U*} : C(U) \to C$ such that $j_{U*}j_U^*M = \underline{\mathrm{Hom}}_C([1 \to A], M)$ for all $M \in C$. Here $[1 \to A]$ denotes the fiber of the map $1 \to A$.
(3) Upper shriek functors: $i_Z^! : C \to C(Z), M \mapsto \underline{\mathrm{Hom}}_C(A, M)$ and $j_U^! := j_U^*$.
(4) Lower shriek functors: $i_{Z!} := i_{Z*}$ and $j_{U!} : C(U) \to C$ such that $j_{U!}j_U^*M = [1 \to A] \otimes M$.

A symmetric monoidal colimit preserving functor $f^* : C \to D$ of presentably closed symmetric monoidal stable ∞-categories is called an open (resp. closed) immersion if it is equivalent to a functor of the form $j_U^* : C \to C(U)$ (resp. $i_Z^* : C \to C(Z)$).

The above functors satisfy the usual adjunction properties. For $M \in C$ and $Z \in \mathcal{S}(C), U = Z^c$, we get the following localization sequence for free:

$$\begin{aligned} &j_{U!}j_U^*M \to M \to i_{Z*}i_Z^*M \to, \\ &i_{Z*}i_Z^!M \to M \to j_{U*}j_U^*M \to . \end{aligned} \tag{2.1}$$

Definition 2.2 ([CS22, Def. 7.1])**.** A categorified locale is a triple (X, C, f) consisting of a locale X, a presentably symmetric monoidal stable ∞-category C and a morphism of locale $f : \mathcal{S}(C) \to X$. A morphism of categorified locales $F : (X, C, f) \to (Y, D, g)$ consists of a morphism of locale $F : X \to Y$ and a colimit preserving exact symmetric monoidal functor $G : D \to C$ commuting with f and g. It can be enriched into an ∞-category (See [Aok25, Def. 4.2]).

Let (A, A^+) be a sheafy Tate Huber pair and $X = \mathrm{Spa}(A, A^+)^{\mathrm{op}}$ the locale of open subspaces of $\mathrm{Spa}(A, A^+)$. By [RC24, Prop. 2.3.2], we can form a categorified locale $(X, D((A, A^+)_\square), f)$ as following: For each $U \in X$ a rational localization of $\mathrm{Spa}(A, A^+)$, the base change functor $j_U^* : D((A, A^+)_\square) \to D((\mathcal{O}(U), \mathcal{O}^+(U))_\square)$ is an open localization. So it corresponds to an idempotent algebra $A_Z \in D((A, A^+)_\square)$. The map f is given by $U \mapsto Z^c \in \mathcal{S}(D((A, A^+)_\square))$. This can be globalized to a fully faithful functor $X \mapsto X^{lo} = (|X|, D(X), f)$ from the 1-category of analytic adic spaces over $\mathbb{Q}_p$ to the ∞-category of categorified locales [RC24, Cor. 2.3.4]: for an analytic adic space X with $U_\bullet$ an hypercover of X by open affinoid subspaces, then

$$D(X) \simeq \varprojlim_{[n] \in \Delta} D(U_n) \tag{2.2}$$

which is independent of the choice of the hypercover.

Lemma 2.3. *Assume X is an analytic adic space, U is an open subset of X and Z is a closed subset of X.*

(1) If $Z \subseteq U$ then there is a natural map $j_{U*}1_U \to A_Z$ in $D(X)$.
(2) If $U \subseteq Z$ then there is a natural map $A_Z \to j_{U*}1_U$ in $D(X)$.

Proof. Let A_Z, A_{U^c} be the idempotent algebra in $\mathcal{S}(D(X))^{\mathrm{op}}$ corresponding to the closed subspaces Z and U^c.

(1) We need to give a natural map

$$\underline{\mathrm{Hom}}([1 \to A_{U^c}], 1) \to A_Z.$$

Since $Z \subseteq U$, we have $A_Z \otimes A_{U^c} = 0$ [CS22, Prop. 5.3(1)] and we get natural maps

$$\underline{\mathrm{Hom}}([1 \to A_{U^c}], 1) \to \underline{\mathrm{Hom}}([1 \to A_{U^c}], A_Z) \simeq \underline{\mathrm{Hom}}_{A_Z}([1 \to A_{U^c}] \otimes A_Z, A_Z) \simeq A_Z,$$

where the first map is induced by the map $1 \to A_Z$ corresponding to $Z \subset U$ and for the last isomorphism we used the assumption that $\otimes$ is colimit preserving and exact.

(2) We need to give a natural map

$$A_Z \to \underline{\mathrm{Hom}}([1 \to A_{U^c}], 1).$$

By adjunction, it suffices to give a natural map $[1 \to A_{U^c}] \otimes A_Z \to 1$. Since $U \subseteq Z$, this implies $[1 \to A_{U^c}] \otimes [1 \to A_Z] = 0$ by [CS22, Prop. 5.3(3)] which means $[1 \to A_{U^c}] \otimes A_Z \simeq [1 \to A_{U^c}]$. Recall that $[1 \to A_{U^c}] = j_{U!}j_U^*1$. By adjunction there is a natural map $[1 \to A_{U^c}] \to 1$. □

Remark 2.4. Suppose that we have inclusions $U_1 \subseteq Z \subseteq U_2 \subset X$ with open immersions $j_{U_1} : U_1 \to X$ and $j_{U_2} : U_2 \to X$ and $j_{12} : U_1 \to U_2$. A diagram chasing shows that the natural map $j_{U_2*}1_{U_2} \to j_{U_1*}1_{U_1}$, coming from the natural map $1_{U_2} \to j_{12*}j_{12}^*1_{U_2}$ and $j_{U_1*}1_{U_1} = j_{U_2*}j_{12*}j_{12}^*1_{U_2}$, factors as $j_{U_2*}1_{U_2} \to A_Z \to j_{U_1*}1_{U_1}$.

The categorified locale associated to an analytic adic space is a derived Tate adic space in [RC24, § 2] which we recall below. We first recall the definition of bounded affinoid analytic rings. Let $\mathrm{AnRing}_{Z_\square}$ be the ∞-category of analytic rings over $\mathbb{Z}_\square$, see [RC24, § 1].

Definition 2.5 ([RC24, Def. 2.6.1, Def. 2.6.6, Def. 2.6.10] or [RC26c, Def. 7.1.1])**.** Let $A = (\underline{A}, D(A)) \in \mathrm{AnRing}_{Z_\square}$. Let $R_\square := (\mathbb{Z}((\pi)), \mathbb{Z}[[\pi]])_\square$ be the analytic ring associated to the Huber pair and let $\mathrm{AniAlg}_{R_\square}$ be the ∞-category of animated $R_\square$-algebras.

(1) The subring of solid elements is a discrete animated ring defined by
$$A^+ := \mathrm{Map}_{\mathrm{AnRing}_{Z_\square}}(\mathbb{Z}[T]_\square, A).$$
(2) The analytic ring A is called an affinoid analytic ring if the induced map $(\underline{A}, \pi_0(A^+))_\square \to A$ is an equivalence of analytic rings.
(3) Assume further that A is over $R_\square$, the condensed subring of power bounded elements is defined as for $S \in \mathrm{ExtDisc}$,
$$A^\circ(S) := \mathrm{Map}_{\mathrm{AniAlg}_{R_\square}}(R_\square\langle\mathbb{N}[S]\rangle, A)$$
and the condensed subring of bounded elements is defined as $A^b := (A^\circ)[1/\pi]$.
(4) Assume A is over $R_\square$. Then A is called bounded if the natural map $A^b \to \underline{A}$ is an equivalence.

The category of bounded affinoid analytic rings over $\mathbb{Q}_p$ is denoted by $\mathrm{AffRing}^b_{\mathbb{Q}_p}$. Let $\mathrm{Aff}^b_{\mathbb{Q}_p}$ be the opposite category of $\mathrm{AffRing}^b_{\mathbb{Q}_p}$. For $A \in \mathrm{AffRing}^b_{\mathbb{Q}_p}$, the corresponding object in $\mathrm{Aff}^b_{\mathbb{Q}_p}$ is denoted by $\mathrm{AnSpec}\, A$ and is called the analytic spectrun of A.

Remark 2.6. We will only consider analytic rings over $\mathbb{Q}_p$. This avoids the discussion between E_∞-rings and commutative animated rings (See [RC24, Rem. 2.1.4]).

Let $A \in \mathrm{AffRing}^b_{\mathbb{Q}_p}$, by [RC24, Thm. 2.7.2, Def. 2.7.3], there is a categorified locale

$$(|\operatorname{Spa} A|, D(A), \rho_A)$$

associated to it. We note that if $A = (R, R^+)_\square$ is associated to a Tate-Huber pair, then $|\operatorname{Spa} A| \cong |\operatorname{Spa}(R, R^+)|$ ([RC24, Prop. 2.7.8(3)]). The category $\mathrm{AffRing}^b_{\mathbb{Q}_p}$ is endowed with the analytic topology given by [RC24, Def. 2.7.20], which is subcanonical ([RC24, Lem. 2.7.21]).

Let $\mathrm{Sh}_{\mathrm{an}}(\mathrm{Aff}^b_{\mathbb{Q}_p})$ be the category of sheaves of anima on the site $\mathrm{Aff}^b_{\mathbb{Q}_p} = \mathrm{AffRing}^{b,\mathrm{op}}_{\mathbb{Q}_p}$ with respect to the analytic topology.

Definition 2.7 ([RC24, Def. 2.7.22])**.** A derived Tate adic space (over $\mathbb{Q}_p$) is a sheaf $X \in \mathrm{Sh}_{\mathrm{an}}(\mathrm{Aff}^b_{\mathbb{Q}_p})$ which admits an open analytic cover by representable sheaves.

Example 2.8. Let X be an analytic adic space over $\mathbb{Q}_p$, we can and will (implicitly) view it as a derived Tate adic space as the following: Let $\mathrm{Affadic}_{\mathbb{Q}_p}$ be the site of analytic affinoid adic spaces over $\mathbb{Q}_p$ endowed with analytic topology. By [RC24, Prop. 2.7.8] and [And21, Prop. 4.12], the functor

$$\mathrm{Affadic}_{\mathbb{Q}_p} \to \mathrm{Aff}^b_{\mathbb{Q}_p}; \mathrm{Spa}(A, A^+) \mapsto \mathrm{AnSpec}((A, A^+)_\square) \tag{2.3}$$

is a continuous functor between sites. Thus it extends to a colimit preserving functor $(-)_{\mathrm{Tate}} : \mathrm{Sh}_{\mathrm{an}}(\mathrm{Affadic}_{\mathbb{Q}_p}) \to \mathrm{Sh}_{\mathrm{an}}(\mathrm{Aff}^b_{\mathbb{Q}_p})$ (i.e. left Kan extension). We view X as an object in $\mathrm{Sh}_{\mathrm{an}}(\mathrm{Affadic}_{\mathbb{Q}_p})$ and define the derived Tate adic space associated to it to be X_{Tate}.

By [RC24, Def. 2.7.22], one can associate a categorified locale $(X_{\mathrm{Tate}})^{lo} = (|X_{\mathrm{Tate}}|, D(X_{\mathrm{Tate}}))$ to X_{Tate} and by [RC24, Cor. 2.7.23] we have an isomorphism of categorified locales $(X_{\mathrm{Tate}})^{lo} \cong X^{lo}$, where $D(X)$ is given in (2.2).

Lemma 2.9. *Let X be an analytic adic space. Assume $Z \subseteq X$ is a closed subset such that $Z = \cap_i U_i$ where $\{U_i\}$ is a cofiltered family of open subsets of X and for each i there is $i' > i$ with Z_i closed such that $U_{i'} \subseteq Z_i \subseteq U_i$. Then $A_Z = \mathrm{colim}_i\, j_{i*}1_{U_i}$. Here A_Z is the idempotent algebra in $D(X)$ corresponding to Z.*

Proof. By [CS22, Prop. 5.3(4)], $A_Z = \mathrm{colim}_i\, A_{Z_i}$ and the condition says that the system $\{A_{Z_i}\}_i$ and $j_{i*}1_{U_i}$ are cofinal to each other (see Lemma 2.3). The conclusion follows. □

2.2. **Six functor formalism and descent for quasi-coherent sheaves.** To have Serre duality, we need a six functor formalism of quasi-coherent sheaves on analytic adic spaces. We first recall a general setup of six functor formalism in [Man22, § A.5] (see also a summary in [RC24, § 3.1]).

A geometric setup [Man22, Def. A.5.1] is a pair $(\mathcal{C}, E)$ consisting of an ∞-category $\mathcal{C}$ with finite limits and E a collection of homotopy classes of morphisms in $\mathcal{C}$ containing all isomorphisms and stable under pullbacks and compositions. Let $\mathrm{Corr}(\mathcal{C}, E)$ be the ∞-category of correspondences associated to $(\mathcal{C}, E)$ [Man22, Def. A.5.2]. Objects of $\mathrm{Corr}(\mathcal{C}, E)$ are those of $\mathcal{C}$, and a morphism from X to Y is a correspondence

$$\begin{array}{ccccc} & & V & & \\ & \swarrow^{f} & & \searrow^{g} & \\ X & & & & Y \end{array} \tag{2.4}$$

in $\mathcal{C}$ with $g \in E$.

A three-functor formalism [Man22, Def. A.5.6] on $(\mathcal{C}, E)$ is a lax symmetric monoidal functor

$$D : \mathrm{Corr}(\mathcal{C}, E) \to \mathrm{Cat}_\infty,$$

where Cat_∞ is the ∞-category of ∞-categories with the product symmetric monoidal structure. Taking f or g to be the identity in (2.4) one obtains functors $g_! : D(X) \to D(Y)$ and $f^* : D(X) \to D(Y)$ respectively. A six-functor formalism [Man22, Def. A.5.7] is a three-functor formalism for which f^*, $f_!$ and $\otimes$ admit right adjoints, denoted by f_*, $f^!$ and $\underline{\mathrm{Hom}}$ respectively.

To construct three or six functor formalisms, [Man22, Def. A.5.9] introduced the notion of a suitable decomposition of E, which consists of a pair $I, P \subset E$ satisfying analogue properties of the classes of étale morphisms and proper morphisms, and every $f \in E$ is of the form $f = p \circ i$ for some $i \in I$ and $p \in P$. Let $\mathcal{C} = \mathrm{Aff}^b_{\mathbb{Q}_p} = \mathrm{AffRing}^{b,\mathrm{op}}_{\mathbb{Q}_p}$.

Definition 2.10 ([RC24, Def. 3.2.2])**.** We have the following classes of morphisms in $\mathcal{C}$:

1. Let I be the class of morphisms $f : \mathrm{AnSpec}\, B \to \mathrm{AnSpec}\, A$ such that $f^* : D(A) \to D(B)$ is an open localization in the sense of Definition 2.1;
2. Let P be the class of morphisms $f : \mathrm{AnSpec}\, B \to \mathrm{AnSpec}\, A$ such that the analytic structure of B is induced from A (i.e. $D(B) = \mathrm{Mod}_{\underline{B}}(D(A))$, the $\underline{B}$-module objects in $D(A)$);
3. Let E be the class of morphisms of the form $p \circ j$ with $j \in I$ and $p \in P$.

By [RC24, Lem. 3.2.5], (I, P) gives a suitable decomposition of E. The functor $D : \mathcal{C}^{\mathrm{op}} \to \mathcal{P}r^L : \mathrm{AnSpec}\, A \mapsto D(A)$ enhances to a six-functor formalism

$$D : \mathrm{Corr}(\mathcal{C}, E) \to \mathcal{P}r^L. \tag{2.5}$$

applying [Man22, Prop. A.5.10]. Here $\mathcal{P}r^L$ denotes the ∞-category of presentable stable ∞-categories and colimit-preserving functors.

Example 2.11. Let $f : (A, A^+) \to (B, B^+)$ be a morphism of analytic Huber pairs.

(1) If f is an open immersion, then f lies in the class I by [RC24, Prop. 2.3.2].
(2) The map f factors through $f' : (A, A^+) \to (B, B^+_A) := (B, \widetilde{(f(A^+) + B^{\circ\circ})})$, where $\widetilde{(-)}$ denotes the normalization. By [And21, Prop. 3.32], one has $(B, B^+_A)_\square \simeq (B, f(A^+))_\square$ whose analytic ring structure is induced from $(A, A^+)_\square$ via $A \to B$. Hence the map of affinoid analytic spaces induced by f' lies in the class P.

Remark 2.12. The map in the class P in the above example is related to Huber's universal compactification of adic spaces [Hub96, Def. 5.1.1]. Let $f : X \to Y$ be a morphism between analytic adic spaces. The universal compactification $\overline{X}^{/Y}$ of f satisfies that for any maps $X \to X' \xrightarrow{f'} Y$ with f' partially proper, the map $X \to X'$ factors through $\overline{X}^{/Y}$, which exists if f is a separated and taut morphisms between rigid analytic spaces [Hub96, Thm. 5.1.5]. If $X = \mathrm{Spa}(B, B^+) \to Y = \mathrm{Spa}(A, A^+)$ is a map between affinoid rigid spaces, then $\overline{X}^{/Y} = \mathrm{Spa}(B, B^+_A)$ where the latter is as in Example 2.11.

There is a D-topology on $\mathcal{C}$ defined as the following:

Definition 2.13 ([Sch25, Def. 5.13])**.** Assume $(\mathcal{C}, E, D)$ is a six functor formalism. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. Let $\{X^{\bullet/Y}\}_{\bullet\in\Delta}$ be the Čech nerve of the morphism f. For any morphism $Y' \to Y$ in $\mathcal{C}$, let $f' : X' := X \times_Y Y' \to Y'$ be the pullback of f.

(1) The map f forms a canonical cover if for any $Z \in \mathcal{C}$ and any $Y' \to Y$ in $\mathcal{C}$, the functor $\mathrm{Hom}_{\mathcal{C}}(-, Z)$ satisfies descent along f', i.e. the following map is an equivalence
$$\mathrm{Hom}_{\mathcal{C}}(Y', Z) \xrightarrow{\simeq} \lim_{\bullet\in\Delta} \mathrm{Hom}_{\mathcal{C}}(X'^{\bullet/Y'}, Z).$$
(2) The map f satisfies universal $*$-descent if for any $Y' \to Y$ in $\mathcal{C}$, the functor D^* satisfies descent along pullback morphism f', i.e. the following map is an equivalence of categories where transition maps are given by $*$-pullback
$$D(Y') \xrightarrow{\simeq} \lim_{\bullet\in\Delta} D^*(X'^{\bullet/Y'}).$$
(3) Assume $f \in E$. The map f satisfies universal !-descent if for any $Y' \to Y$ in $\mathcal{C}$, the functor $D^!$ satisfies descent along pullback morphism f', i.e. the following map is an equivalence of categories where transition maps are given by !-pullback
$$D(Y') \xrightarrow{\simeq} \lim_{\bullet\in\Delta} D^!(X'^{\bullet/Y'}).$$
(4) The map f is a D-cover if it is a canonical cover, satisfying universal $*$-descent and universal !-descent. The D-topology on $\mathcal{C}$ is generated by D-covers.

Remark 2.14. Assume $f : X \to Y$ lies in E and satisfies !-descent. Let $\mathcal{F} \in D(Y)$, we have an equivalence in $D(Y)$
$$\mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}} f_{n!} f_n^! \mathcal{F} \xrightarrow{\sim} \mathcal{F}$$
where $f_n : X^{n/Y} \to Y$. This is because for $\mathcal{G} \in D(Y)$ we have
$$\mathrm{Hom}_{D(Y)}(\mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}} f_{n!} f_n^! \mathcal{F}, \mathcal{G}) \simeq \lim_{[n]\in\Delta} \mathrm{Hom}_{D(X^{n/Y})}(f_n^! \mathcal{F}, f_n^! \mathcal{G}) \simeq \mathrm{Hom}_{D(Y)}(\mathcal{F}, \mathcal{G}).$$
Let $f : X \to Y$ be a morphism in $\mathcal{C}$ and satisfies $*$-desecent. Assume $D(Y)$ is complete and $\mathcal{F} \in D(Y)$, the same argument as above shows that we have an equivalence in $D(Y)$:
$$\mathcal{F} \xrightarrow{\sim} \lim_{[n]\in\Delta} f_{n*} f_n^* \mathcal{F}.$$

From now on, the $*$- and !-descent will refer to the above definition and remark.

Definition 2.15 ([RC24, Def. 3.2.10])**.** The ∞-category of Tate stacks over $\mathbb{Q}_p$ is the category $\mathrm{Sh}_D(\mathrm{Aff}^b_{\mathbb{Q}_p})$ of sheaves in anima of $\mathrm{Aff}^b_{\mathbb{Q}_p}$ for the D-topology.

By [HM24, Theorem 3.4.11], we can extend the six functor formalism (2.5) to $(\mathrm{Sh}_{\mathrm{an}}(\mathcal{C}), E') = (\mathrm{Sh}_{\mathrm{an}}(\mathrm{Aff}^b_{\mathbb{Q}_p}), E')$ and $(\mathrm{Sh}_D(\mathcal{C}), E'')$ where E' (resp. E'') is a collection of morphisms in $\mathrm{Sh}_{\mathrm{an}}(\mathcal{C})$ (resp. $\mathrm{Sh}_D(\mathcal{C})$) containing E, local on the target and source (cf. [RC24, Cor. 3.2.6]). We call morphisms in E' (resp. E'') !-able morphisms.

Example 2.16. Let X be an analytic adic space over $\mathbb{Q}_p$, we can view it as a Tate stack as the following (See Example 2.8 and [Mik26a, § 1.3]): Let $\mathrm{Affadic}_{\mathbb{Q}_p}$ be the site of analytic affinoid adic spaces over $\mathbb{Q}_p$ endowed with analytic topology. By [And21, Prop. 4.12] and [RC24, Lem. 3.2.9], the functor (2.3) is also a continuous functor where $\mathrm{Aff}^b_{\mathbb{Q}_p}$ is now endowed with D-topology. Thus it extends to a colimit preserving functor $(-)_\square : \mathrm{Sh}_{\mathrm{an}}(\mathrm{Affadic}_{\mathbb{Q}_p}) \to \mathrm{Sh}_D(\mathrm{Aff}^b_{\mathbb{Q}_p})$. We view X as an object in $\mathrm{Sh}_{\mathrm{an}}(\mathrm{Affadic}_{\mathbb{Q}_p})$ and define the Tate stack associated to it to be $X_\square \in \mathrm{Sh}_D(\mathrm{Aff}^b_{\mathbb{Q}_p})$.

If $\{j_i : U_i \to X\}_{i\in I}$ is an analytic cover of X, then $\{j_{i\square} : U_{i\square} \to X_\square\}_{i\in I}$ is a D-cover: Put $U = \sqcup_i U_i$. We note that by [Mik26a, Exam. 1.55], we have $(U_i \times_X U_j)_\square \simeq U_{i\square} \times_{X_\square} U_{j\square}$, and similarly for higher intersections. Thus, by construction and colimit preservation, $X_\square \simeq \mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}} U^{n/X_\square}_\square$, so $\{j_{i\square} : U_{i\square} \to X_\square\}_{i\in I}$ is a canonical cover. By [Mik26a, Rem. 1.15] (cf. Lem. 1.56 there), it satisfies universal D^*-descent. In particular, as (2.2),

$$D(X) = \varprojlim_{[n]\in\Delta} D^*(U^{n/X}).$$

By [Mik26a, Lem. 1.57], the map $j_{i\square}$ is D-étale [HM24, Def. 4.6.1]. Hence by [HM24, Lem. 4.6.4(i)] we have $j^!_{i\square} = j^*_{i\square}$ (see also [MW24, § 4.7]). So $\{j_{i\square} : U_{i\square} \to X_\square\}_{i\in I}$ also satisfies universal $D^!$-descent by [HM24, Lem. 4.7.1].

Remark 2.17. By !-descent in the above example, cohomology of an analytic adic space does not change whenever we view analytic adic spaces as analytic sheaves or D-sheaves, or a solid stack [RC24, Def. 3.2.7]. We choose to use analytic topology because it is not clear if a derived Tate adic space is a Tate stack and we need the locale in our proof of comparison of two compactly supported cohomologies in § 2.3, where the topology spaces matter. Note that, a priori, the closed subsets $Z \subset X$ in Lemma 2.9 or Lemma 2.3 need not be adic spaces nor analytic stacks.

All discussions in [RC24, § 3.6] are on analytic rings, thus apply in our setting and we can deduce the Serre duality. Let E be a finite extension of $\mathbb{Q}_p$. Let $f : X \to * = \mathrm{Spa}(E, \mathcal{O}_E)$ be a smooth rigid analytic spaces over E. Then the above six functor formalism supplies a functor $f_! : D(X) \to D(E_\square)$. Let ω_X be the dualizing complex of X. For $\mathcal{F} \in D(X)$, we write

$$R\Gamma_c(X, \mathcal{F}) := f_!\mathcal{F}$$

and $R\mathscr{H}om(\mathcal{F}, \omega_X)$ for the internal hom in $D(X)$, where $\omega_X = f^!E$.

Theorem 2.18 ([RC24, Thm. 3.6.15])**.** *Assume X is a smooth rigid analytic variety over E and $\mathcal{F} \in D(X)$ is a quasi-coherent sheaf on X, then we have a quasi-isomorphism*

$$R\underline{\mathrm{Hom}}_E\left(R\Gamma_c(X, \mathcal{F}), E\right) \simeq R\Gamma(X, R\mathscr{H}om(\mathcal{F}, \omega_X))$$

in $D(E_\square)$.

Remark 2.19. We remark on the group actions on (compactly supported) cohomologies and duality. Assume G is a p-adic Lie group acting continuously on a smooth rigid analytic variety X and $\mathcal{F}$ is a G-equivariant vector bundle (see § 3.2 for more discussions). For each $g \in G$ we have an isomorphism $\sigma_g : g^*\mathcal{F} \simeq \mathcal{F}$. It induces the following maps:

$$\tau_g : R\Gamma(X, \mathcal{F}) \xrightarrow{\simeq} R\Gamma(X, g_*g^*\mathcal{F}) \simeq R\Gamma(X, g^*\mathcal{F}) \xrightarrow{\sigma_g} R\Gamma(X, \mathcal{F}),$$

$$R\Gamma_c(X, \mathcal{F}) \xleftarrow{\simeq} R\Gamma_c(X, g_!g^!\mathcal{F}) \simeq R\Gamma_c(X, g^*\mathcal{F}) \xleftarrow{\sigma_g^{-1}} R\Gamma_c(X, \mathcal{F}) : \tau_{g,c}.$$

Here we used that $g_* \simeq g_!, g^* \simeq g^!, g_*g^* \simeq \mathrm{id}$ and $g_!g^! \simeq \mathrm{id}$, and the middle isomorphisms are induced by the isomorphism $g\cdot : g^{-1}X \xrightarrow{\sim} X$. This tells that the Serre duality is compatible with the G-actions.

2.3. **Comparison of two compactly supported cohomologies.** Let X be a partially proper rigid analytic variety over $\mathbb{Q}_p$. For a vector bundle $\mathcal{F}$ on X, we can view it as an object in $D(X)$ and define the (compactly supported) cohomology $R\Gamma_{(c)}^{\mathrm{CS}}(X,\mathcal{F})$ using the six functor formalism in the last subsection. On the other hand we can view $\mathcal{F}$ as a solid abelian sheaf on X and define the (compactly supported) cohomology $R\Gamma_{(c)}^{cl}(X,\mathcal{F})$ for the analytic topology of X [AL25] (for condensed version see [LSW26, Appendix A]). Here the superscripts "CS" and "cl" signify "Clausen–Scholze" and "classical" respectively.

In this subsection, we are going to show that these two compactly supported cohomologies coincide (Corollary 2.24).

Remark 2.20. A quicker way to compare two compactly supported cohomologies (in smooth case) is to use the Serre duality in both settings (for classical setting see [AGN25, § 5]). But we prefer to give a direct proof.

The classical compactly supported cohomology has the following property:

Proposition 2.21. *Suppose X is a partially proper rigid analytic variety and $\{U_\alpha\}$ is a locally finite covering (i.e. for any $x\in X$, there exists an open neighbourhood U_x such that $U_x\cap U_\alpha=\emptyset$ for all but finitely many α) of X by affinoid open subspaces. For a vector bundle $\mathcal{F}$ on X, we have naturally*

$$R\Gamma_c^{cl}(X,\prod_\alpha j_{\alpha*}j_\alpha^{-1}\mathcal{F})\simeq\oplus_\alpha R\Gamma^{cl}(U_\alpha,j_\alpha^{-1}\mathcal{F}),$$

where $j_\alpha:U_\alpha\hookrightarrow X$ is the open immersion and $j_{\alpha*}$ denotes the non-derived pushforward.

Proof. Note that partially proper adic spaces are taut so that the closure of U_α in X is still quasi-compact. Sicne the covering is locally finite, the direct product $\prod_\alpha j_{\alpha*}j_\alpha^{-1}\mathcal{F}$ is (locally) a direct sum. By an argument on the compactness of the support and [LSW26, Def. A.15], it is not hard to see that for any abelian sheaf $\mathcal{F}_\alpha$ on U_α, we have

$$\Gamma_c(X,\prod j_{\alpha*}\mathcal{F}_\alpha)\simeq\oplus_\alpha\Gamma(U_\alpha,\mathcal{F}_\alpha).$$

Choose an injective resolution of $j_\alpha^{-1}\mathcal{F}\to\mathcal{I}_\alpha^\bullet$. Since j^{-1} is exact, we know $j_{\alpha*}\mathcal{I}_\alpha^\bullet$ is a complex of injective sheaves. Moreover, by Tate's acyclicity and our convention that X is separated, we have $H^i(V\cap U_\alpha,j_\alpha^{-1}\mathcal{F})=0$ for any affinoid open $V\subset X$ and $i\geq 1$. Hence $Rj_{\alpha*}\mathcal{F}=j_{\alpha*}\mathcal{F}$. Consequently, $j_{\alpha*}j_\alpha^{-1}\mathcal{F}\to j_{\alpha*}\mathcal{I}_\alpha^\bullet$ is an injective resolution. Therefore we have

$$R\Gamma_c^{cl}(X,\prod j_{\alpha*}j_\alpha^{-1}\mathcal{F})\simeq\Gamma_c(X,\prod j_{\alpha*}\mathcal{I}_\alpha^\bullet)\simeq\oplus_\alpha\Gamma(U_\alpha,\mathcal{I}_\alpha^\bullet)\simeq\oplus_\alpha R\Gamma(U_\alpha,j_\alpha^{-1}\mathcal{F}).$$

The result follows. □

Lemma 2.22. *Let X be a separated analytic adic space over $\mathbb{Q}_p$ and $\mathcal{F}$ be a vector bundle over X. We have a natural quasi-isomorphism of solid $\mathbb{Q}_p$-vector spaces*

$$R\Gamma^{\mathrm{CS}}(X,\mathcal{F})\simeq R\Gamma^{cl}(X,\mathcal{F}),$$

where we view $\mathcal{F}\in D(X)$ for the left-hand side and view $\mathcal{F}$ as a solid abelian sheaf on $|X|$ for the right-hand side.

Proof. Let $\{U_i=\mathrm{Spa}(A_i,A_i^+)\}_{i\in I}$ be an open covering of X by affinoid subspaces such that $\mathcal{F}|_{U_i}$ is free. Let $Y=\sqcup_{i\in I}U_i$. By $*$-descent (Example 2.16), $R\Gamma^{\mathrm{CS}}(X,\mathcal{F})$ is computed by

$$\lim_{[n]\in\Delta}\mathcal{F}(Y^{n/X}) \tag{2.6}$$

where the index is the Čech nerve of the open covering $Y\to X$. However, by Čech to derived comparison ([Sta25, Tag 03F9]) and Tate's acyclicity $R\Gamma^{cl}(U_i,\mathcal{F}|_{U_i})=H^0(U_i,\mathcal{F}|_{U_i})[0]$, the classical cohomology $R\Gamma^{cl}(X,\mathcal{F})$ is also computed by the complex (2.6). □

To compare compactly supported cohomology, we first consider local cases.

Proposition 2.23. *Let $X=\mathrm{Spa}(A,A^\circ)$ be an affinoid rigid analytic variety and let $j:X^0\subseteq X$ be an open immersion such that X^0 is a Stein variety (we call such an X^0 a naive interior of X). Assume $\mathcal{F}$ is a vector bundle on X. Then we have*

$$R\Gamma_c^{\mathrm{CS}}(X^0,\mathcal{F})\simeq R\Gamma_c^{cl}(X^0,\mathcal{F}).$$

Proof. Suppose $X^0 = \cup_{i\geq 0} V_i$ where $V_i = \mathrm{Spa}(A_i, A_i^+)$ form a system of strict inclusions of open affinoids. Let $j_i : V_i \to X$ be the open immersion. Let $Z := |X \backslash X^0|$ be the closed subset of X. Let $\overline{V}_i$ be the closure of V_i. Then $Z = \cap_i |X \backslash \overline{V_i}|$ satisfies the condition in Lemma 2.9. Let $W_i = X \backslash \overline{V_i}$, $l_i : W_i \to X$ and $i : Z \to X$.

Now we view these objects as open or closed objects in X^{lo}. The map $V_i \to V_{i+1}$ factors through $V_i^{cp} = \mathrm{AnSpec}(A_i, \mathbb{Z})$ (actually the universal compactification of $V_i \to *$ is homeomorphic to $\overline{V}_i$ since X^0 is partially proper, cf. [ALY23, Prop. 3.4.5(1)]), and likewise so does j_i, and we have a decomposition (see Remark 2.12)

$$V_i \xrightarrow{j_i'} V_i^{cp} \xrightarrow{j_i''} X \xrightarrow{f} *.$$

Since j_i'' lies in P, we have $j_{i*}'' = j_{i!}''$, and hence

$$R\Gamma_c^{\mathrm{CS}}(V_i, j_i^* \mathcal{F}) = (f j_i'')_* j_{i!}' j_i^* \mathcal{F} = f_*(j_i'' j_i')_! j_i^* \mathcal{F} = R\Gamma^{\mathrm{CS}}(X, j_{i!} j_i^* \mathcal{F}).$$

Let $c_i : Z_i \subseteq X$ be the complement of V_i in X. By (2.1) we have an exact triangle

$$j_{i!} j_i^* \mathcal{F} \to \mathcal{F} \to c_{i*} c_i^* \mathcal{F} \to .$$

Taking global sections on X we get an exact triangle

$$R\Gamma_c^{\mathrm{CS}}(V_i, j_i^* \mathcal{F}) \to R\Gamma^{\mathrm{CS}}(X, \mathcal{F}) \to R\Gamma^{\mathrm{CS}}(Z_i, c_i^* \mathcal{F}) \to .$$

Here we use $R\Gamma^{\mathrm{CS}}(Z_i, c_i^* \mathcal{F})$ to denote $A_{Z_i} \otimes^L_{(A, A^\circ)_\square} M$ as an object in $D(\mathbb{Q}_{p,\square})$, where A_{Z_i} is the idempotent algebra corresponding to Z_i and $M \in D((A, A^\circ)_\square)$ is the complete A-module corresponding to $\mathcal{F}$. Taking the colimit along i and applying Lemma 2.9, we get a triangle

$$R\Gamma_c^{\mathrm{CS}}(X^0, j^* \mathcal{F}) \to R\Gamma^{\mathrm{CS}}(X, \mathcal{F}) \to \mathrm{colim}_i R\Gamma^{\mathrm{CS}}(W_i, l_i^* \mathcal{F}) \to .$$

On the other hand, for the classical compactly supported cohomology, we have $R\Gamma_c^{cl}(X^0, \mathcal{F}) \simeq R\Gamma^{cl}(X, j_! \mathcal{F})$. This is because $\Gamma_c(X^0, \mathcal{F}) \simeq \Gamma(X, j_! \mathcal{F})$; deriving the functor and noting that the (underived) extension-by-zero functor $j_!$ is exact, we get a triangle

$$R\Gamma_c^{cl}(X^0, j^{-1} \mathcal{F}) \to R\Gamma^{cl}(X, \mathcal{F}) \to R\Gamma^{cl}(Z, i^{-1} \mathcal{F}) \to .$$

There is a natural map (similarly to Remark 2.4) $\mathrm{colim}_i Rl_{i*} l_i^{-1} \mathcal{F} \to i_* i^{-1} \mathcal{F}$ which induces (for the first isomorphism below we use that X is quasi-compact)

$$\iota : \mathrm{colim}_i R\Gamma^{cl}(W_i, l_i^{-1} \mathcal{F}) \simeq R\Gamma^{cl}(X, \mathrm{colim}_i Rl_{i*} l_i^{-1} \mathcal{F}) \to R\Gamma^{cl}(Z, i^{-1} \mathcal{F}).$$

Note that $\{W_i\}_i$ is cofinal in the system of open neighbourhoods of Z, so by [AL25, Lem. 2.12] (for the condensed version using [LSW26, Prop. A.13]) the map ι is a quasi-isomorphism. This implies the desired result by Lemma 2.22. □

Corollary 2.24. *Let X be a partially proper rigid analytic variety over $\mathbb{Q}_p$ and let $\mathcal{F}$ be a vector bundle on X. Then*

$$R\Gamma_c^{\mathrm{CS}}(X, \mathcal{F}) \simeq R\Gamma_c^{cl}(X, \mathcal{F}).$$

Proof. By our convention, a partially proper rigid analytic variety X is countable at infinity and can be covered by naive interiors $\{X_i\}_{i\in\mathbb{N}}$ as in Proposition 2.23. By !-descent (Example 2.16), we have

$$R\Gamma_c^{\mathrm{CS}}(X, \mathcal{F}) \simeq \mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}} \oplus_{i_0,\cdots,i_n\in\mathbb{N}} R\Gamma_c^{\mathrm{CS}}(X_{i_0} \times_X \cdots \times_X X_{i_n}, \mathcal{F}).$$

Classical compactly supported cohomology also admits analytic descent. This is because we have a resolution of $\mathcal{F}$ by $\oplus_{i_0,\cdots,i_n} j_{i_0,\cdots,i_n!} j_{i_0,\cdots,i_n}^{-1} \mathcal{F}$ where $j_{i_0,\cdots,i_n} : X_{i_0} \times_X \cdots \times_X X_{i_n} \to X$ (the exactness can be checked on stalks by [LSW26, Cor. A.10], cf. [AL25, § 4.7]). Note that $j_{i!}$ is exact and $R\Gamma_c^{cl}(X, -)$ commutes with direct limits ([AL25, Cor. 4.11], the proof works in solid case, see [LSW26, proof of Lemma A.21]). Hence

$$R\Gamma_c^{cl}(X, \mathcal{F}) \simeq \mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}} \oplus_{i_0,\cdots,i_n\in\mathbb{N}} R\Gamma_c^{cl}(X_{i_0} \times_X \cdots \times_X X_{i_n}, \mathcal{F}).$$

So the result follows from Proposition 2.23. □

2.4. **Two functional analysis results.** In this subsection, we show that compactly supported cohomology of an equidimensional smooth affinoid rigid analytic variety concentrates in one degree and is a Smith space (Proposition 2.28). This does not follow directly from Serre duality (Theorem 2.18). We also collect a result on functional analysis for later usage (Proposition 2.33).

Let E be a finite extension of $\mathbb{Q}_p$. Let $X = \mathrm{Spa}(A, A^\circ)$ be a smooth affinoid rigid analytic variety of pure dimension d over E and $\mathcal{F}$ be a vector bundle on X.

Lemma 2.25. *Assume L is a complex of Smith spaces which concentrates in degree ≤ 0. Assume that $R\underline{\mathrm{Hom}}(L, E)$ has only possible nonzero cohomology in degree zero. Then there is a quasi-isomorphism $B[0] \simeq R\underline{\mathrm{Hom}}(L, E)$ where B is a Banach space over E and $L \simeq S[0]$ where $S = B^\vee := \underline{\mathrm{Hom}}(B, E)$ is a Smith space over E.*

Proof. Since Smith spaces are projective in $\mathrm{Mod}(E_\square)$ [RJRC22, Lem. 3.8], $R\underline{\mathrm{Hom}}(L, E) \simeq L^{-\bullet,\vee}$ can be represented by a complex of Banach spaces concentrating in degree ≥ 0. Let $B := \ker(L^{0,\vee} \to L^{-1,\vee})$ which is a Banach space. By assumption, the complex

$$0 \to B \to L^{0,\vee} \to L^{-1,\vee} \to \cdots \to L^{-s,\vee} \to \cdots$$

is an exact sequence of Banach spaces over E. Since $\underline{\mathrm{Hom}}(-, E)$ induces an exact anti-equivalence between Smith spaces and Banach spaces (cf. [RJRC22, Thm. 3.40(1)]), we get that

$$\cdots \to L^{-s} \to L^{-s+1} \to \cdots L^0 \to B^\vee \to 0$$

is an exact sequence of Smith spaces. □

Let $m \geq 1$ and $T_E := E\langle T_1, \cdots, T_m\rangle$ be the Tate algebra and $R_\mathbb{Z} = \mathbb{Z}[T_1, \cdots, T_m]$. Consider spaces

$$Z = \mathrm{AnSpec}((E\langle T_1, \cdots, T_m\rangle, \mathcal{O}_E\langle T_1, \cdots, T_m\rangle)_\square), \quad \overline{Z} := \mathrm{AnSpec}((E\langle T_1, \cdots, T_m\rangle, \mathbb{Z})_\square). \tag{2.7}$$

Define a $(\mathbb{Z}[T_1, \cdots, T_m], \mathbb{Z})_\square$-module

$$R_\infty := (\mathbb{Z}((T_1^{-1}))/\mathbb{Z}[T_1]) \otimes_{\mathbb{Z}_\square} \cdots \otimes_{\mathbb{Z}_\square} (\mathbb{Z}((T_m^{-1}))/\mathbb{Z}[T_m])$$

and set (cf. [RJRC22, Lem. 2.16])

$$T_E^\vee := (\mathcal{O}_E[[T_1^{-1}, \cdots, T_m^{-1}]])[1/p] \cdot T_1^{-1} \cdots T_m^{-1} \simeq \underline{\mathrm{Hom}}_E(T_E, E).$$

Consider the maps $g : Z \xrightarrow{j} \overline{Z} \xrightarrow{g'} \mathrm{AnSpec}\, E$.

Lemma 2.26. *We have an equivalence of functors $g_!(-) \simeq g'_* j_!(-) \simeq - \otimes^L_{(R_\mathbb{Z}, \mathbb{Z})_\square} R_\infty[-m]$ from $D(Z)$ to $D(E_\square)$.*

Proof. We have a pullback diagram

$$\begin{array}{ccc} \mathrm{AnSpec}((T_E, R_\mathbb{Z})_\square) & \longrightarrow & \mathrm{AnSpec}((T_E, \mathbb{Z})_\square) \\ \downarrow & & \downarrow \\ \mathrm{AnSpec}((R_\mathbb{Z}, R_\mathbb{Z})_\square) & \longrightarrow & \mathrm{AnSpec}((R_\mathbb{Z}, \mathbb{Z})_\square) \end{array}$$

where each horizonal arrow is an open immersion in the sense of categorified locale by [Sch19, § 8]. By [And21, Prop. 3.32], $\mathrm{AnSpec}((T_E, R_\mathbb{Z})_\square) \cong Z$. Assume the bottom horizonal open immersion corresponds to the idempotent algebra B, then it is easy to check the top horizontal open immersion corresponds to $B' := B \otimes^L_{(R_\mathbb{Z}, \mathbb{Z})_\square} (T_E, \mathbb{Z})_\square$ (cf. [CS22, Lem. 6.4(4)]). By Definition 2.1 and that $g'_* = g'_!$ is the forgetful functor, for $M \in D(Z)$, we have $g_! M = g'_* j_!(M) = [M \to M \otimes^L_{(T_E, \mathbb{Z})_\square} B'] = [M \to M \otimes^L_{(R_\mathbb{Z}, \mathbb{Z})_\square} B] \simeq h_!(M)$ where $h : \mathrm{AnSpec}((R_\mathbb{Z}, R_\mathbb{Z})_\square) \to \mathrm{AnSpec}\, \mathbb{Z}_\square$. So it suffices to show $h_!(-) \simeq - \otimes^L_{(R_\mathbb{Z}, \mathbb{Z})_\square} R_\infty[-m]$. This is in [Sch19, Thm. 8.13]. □

Lemma 2.27. *We have a quasi-isomorphism in $D((R_\mathbb{Z}, \mathbb{Z})_\square)$:*

$$T_E \otimes^L_{(R_\mathbb{Z}, \mathbb{Z})_\square} R_\infty \simeq (\mathcal{O}_E[[T_1^{-1}, \cdots, T_m^{-1}]])[1/p] \cdot T_1^{-1} \cdots T_m^{-1} = T_E^\vee.$$

Proof. Note that we have

$$T_E \simeq E\langle T\rangle^{\otimes m}, \quad (R_\mathbb{Z}, \mathbb{Z})_\square \sim (\mathbb{Z}[T], \mathbb{Z})_\square^{\otimes m}, \quad T_E^\vee \simeq (\mathcal{O}_E[[T^{-1}]][\frac{1}{p}] \cdot T^{-1})^{\otimes m}.$$

So it suffices to show the case $m = 1$. We have a resolution

$$0 \longrightarrow \mathbb{Z}[[X]][T] \xrightarrow{XT-1} \mathbb{Z}[[X]][T] \longrightarrow \mathbb{Z}((T^{-1})) \longrightarrow 0.$$

One computes that (using that $E\langle T\rangle = \mathcal{O}_E\langle T\rangle[1/p]$, $\otimes^L$ commutes with colimits and [Sch19, Prop. 6.3]):

$$E\langle T\rangle \otimes^L_{(\mathbb{Z}[T],\mathbb{Z})_\square} \mathbb{Z}[[X]][T] \simeq E\langle T\rangle \otimes^L_{\mathbb{Z}_\square} \mathbb{Z}[[X]] \simeq \mathcal{O}_E[[X]]\langle T\rangle[1/p]$$

Thus $E\langle T\rangle \otimes^L_{(\mathbb{Z}[T],\mathbb{Z})_\square} (\mathbb{Z}((T^{-1}))/\mathbb{Z}[T]) \simeq \mathrm{Cone}(E\langle T\rangle \to \mathcal{O}_E[[T^{-1}]]\langle T\rangle[1/p]) \simeq \mathcal{O}_E[[T^{-1}]]\cdot T^{-1}[1/p]$. □

Proposition 2.28. *Let X be a smooth affinoid rigid variety of pure dimension d. We have*

$$R\Gamma_c(X,\mathcal{F}) \simeq H^d_c(X,\mathcal{F})[-d]$$

and $H^d_c(X,\mathcal{F})$ is a Smith space over E.

Proof. First, it suffices to show $R\Gamma_c(X,\mathcal{F})$ can be represented by a complex of Smith spaces concentrated in degree $\leq d$: By Serre duality (Theorem 2.18), the dual of $R\Gamma_c(X,\mathcal{F})[d]$ is quasi-isomorphic to a Banach space concentrated in degree 0. Then by Lemma 2.25, $R\Gamma_c(X,\mathcal{F})[d]$ is quasi-isomorphic to a Smith space concentrate in degree 0.

If the statement holds for the structure sheaf $\mathcal{O}$, since $\mathcal{F}$ can be resolved by sheaves of the form $\mathcal{O}^{\oplus m}$, we find $R\Gamma_c(X,\mathcal{F})$ can be represented by a complex of Smith spaces concentrated in degree $\leq d$. So it suffices to consider the case $\mathcal{F} = \mathcal{O}$.

Now we show that it suffices to assume $A = E\langle T_1,\cdots,T_m\rangle/(f_1,\cdots,f_r)$ where $f_1,\cdots,f_r$ form a regular sequence in $E\langle T_1,\cdots,T_m\rangle$ such that $d = m-r$. We can find finitely many open affinoids $U_1,\cdots,U_n$ covering X such that each U_i are complete intersection. The existence of such cover can be seen using, for example, the Jacobian criterion [Hub96, Prop. 1.7.1, Cor. 1.6.10]. It is clear that a rational subdomain of a complete intersection is a complete intersection thus we can further assume U_i are rational subdomains of X. In particular, any intersections of U_i are also complete intersections since intersections of rational subdomains are rational subdomains [Bos14, § 3.3, Prop. 14]. Consider the open covering $\sqcup_i U_i \to X$. By !-descent and our assumption, we have

$$[H^d_c(U_1\cap\cdots\cap U_n,\mathcal{O}) \to \cdots \to \oplus_{1\leq i<j\leq n} H^d_c(U_i\cap U_j,\mathcal{O}) \to \oplus_{1\leq i\leq n} H^d_c(U_i,\mathcal{O})] \xrightarrow{\simeq} R\Gamma_c(X,\mathcal{O})$$

where the term $\oplus_{1\leq i\leq n} H^d_c(U_i,\mathcal{O})$ lies in degree d. Thus $R\Gamma_c(X,\mathcal{O})$ can be represented by a complex of Smith spaces concentrated in degree $\leq d$ if the statement is proved for complete intersections.

Now assume that $A = E\langle T_1,\cdots,T_m\rangle/(f_1,\cdots,f_r)$ is a complete intersection and $\mathcal{F}$ is free of rank one. Put $X = \mathrm{AnSpec}((A,A^\circ)_\square)$ and set $Z, \overline{Z}$ as in (2.7). We have a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & Z & \xrightarrow{j} & \overline{Z} \\ & \searrow^{f} & \downarrow{\scriptstyle g} & \swarrow_{g'} & \\ & & \mathrm{AnSpec}\, E & & \end{array}$$

where i and g' lie in the class P of Definition 2.10. Hence $R\Gamma_c(X,\mathcal{O}) = f_!\mathcal{O} \simeq g'_* j_! i_* \mathcal{O}$. By Lemma 2.26 and Lemma 2.27, we have

$$g'_* j_! i_* \mathcal{O} \simeq \mathrm{Kos}(f_1,\cdots,f_r;T_E) \otimes^L_{(T_E,\mathbb{Z})_\square} T_E^\vee[-m],$$

where Kos denotes the Koszul complex so that $A \simeq \mathrm{Kos}(f_1,\cdots,f_r;T_E)$ [Sta25, Tag 062F]. The above sequence is formally the dual (via $\underline{\mathrm{Hom}}_E(-,E)$) of $\mathrm{Kos}(f_1,\cdots,f_r;T_E)$ by the self duality of the Koszul complexes [Ser75, § A.2, Ch. IV]. Since $\underline{\mathrm{Hom}}_E(-,E)$ is exact on strict exact sequence of Banach spaces [RJRC22, Prop. 3.5], $\mathrm{Kos}(f_1,\cdots,f_r;T_E) \otimes^L_{(T_E,\mathbb{Z})_\square} T_E^\vee$ is quasi-isomorphic to $\underline{\mathrm{Hom}}_E(A,E)[r]$. We finished the proof. □

Recall the *trace-class functor* ([Bos23a, Def. A.46] and see [Bos23b, Def. A.13] for the derived version)

$$(-)^{\mathrm{tr}} : \mathrm{Mod}(E_\square) \to \mathrm{Mod}(E_\square), W \mapsto (S \in \mathrm{ExtDisc} \mapsto (E_\square[S]^\vee \otimes^\square_E W)(*)). \tag{2.8}$$

If $W \in \mathrm{Mod}(E_\square)$, then a map $P \to W$ from a Smith space P is *trace-class* if and only if the map factors through $W^{\mathrm{tr}} \to W$. The last map is an isomorphism if and only if W is nuclear [RJRC22, Def. 3.14]. See [Bos23a, Lem. A.47] and also [Bos23b, Lem. A.14].

By the flatness in [Bos23a, Prop. A.27], the functor $(-)^{\mathrm{tr}}$ is exact on Smith spaces, and commutes with direct sum. Moreover, if W is Smith, then by the proof of [Bos23a, Lem. A.58], $W^{\mathrm{tr}} = W^B$ is the associated Banach space [RJRC22, Def. 3.34(3)]: choosing a lattice W_0 of W then

$$W^B := \lim_n (\underline{W_0(*)_{\mathrm{dis}}}/\varpi_E^n)[1/\varpi]. \tag{2.9}$$

Corollary 2.29. *Suppose there is a strict open immersion of smooth affinoid rigid analytic varities $j : X_1 \to X_2$ pure of dimension d such that there is a Stein variety X_1^0 with open immersions $X_1 \subset X_1^0 \subset X_2$. Let $\mathcal{F}$ be a vector bundle on X_2. Then the induced map $j_c : R\Gamma_c^{\mathrm{CS}}(X_1, \mathcal{F}) \to R\Gamma_c^{\mathrm{CS}}(X_2, \mathcal{F})$ is a trace-class map in $D(E_\square)$.*

Proof. The compactly supported cohomologies of X_1, X_1^0, and X_2 are concentrated in degree d (by Proposition 2.28), and $H_c^d(X_1, \mathcal{F})$ is a Smith space. After shifting by d, the map j_c factors as

$$j_c : H_c^d(X_1, \mathcal{F}) \to H_c^d(X_1^0, \mathcal{F}) \to H_c^d(X_2, \mathcal{F}).$$

Since X_1^0 is Stein, $H_c^d(X_1^0, \mathcal{F})$ is nuclear by [Bos23a, Cor. A.50, Thm. A.43] being a direct limit of Banach spaces (dual of the Fréchet space $H^0(X_1^0, \mathcal{F}^\vee \otimes \Omega_{X_1^0})$), the first map in the factorization is trace-class, so is j_c by [Bos23a, Rem. A.45]. □

Now we come to the second result, this is a condensed version of the p-adic Schwarz lemma [Kie67a, Satz 1.5].

Definition 2.30 ([RJRC22, Def. 3.33])**.** For two Banach E-vector spaces V, W, a morphism $f : V \to W$ is called compact if the dual map between Smith spaces $f^\vee : W^\vee \to V^\vee$ is trace-class where $V^\vee := \underline{\mathrm{Hom}}(V, E) \in \mathrm{Mod}(E_\square)$.

Remark 2.31. Note that classical compact morphism $f : W \to V$ between Banach spaces is a compact morphism in condensed setting. One can check that in this case, $f^\vee : V^\vee \to W^\vee$ factors through $(W^\vee)^B$.

Lemma 2.32. *Assume $W^\bullet, V^\bullet$ are two bounded complexes of solid Smith E-vector spaces. Suppose we have compatible morphisms $f^i : W^i \to V^i$ which are trace-class. Then regarding $f^\bullet$ as a morphism in $D(E_\square)$, it is trace-class (*[CS22, Def. 8.1]*). In particular, if $f^\bullet$ induces a quasi-isomorphism, then $W^\bullet$ (so is $V^\bullet$) is a perfect complex as an object in $D(E_\square)$.*

Proof. By our assumption, f^i factors as $W^i \xrightarrow{f_1^i} V^{i,B} \xrightarrow{f_2^i} V^i$. It is clear that $V^{\bullet,B}$ form a complex of Banach spaces and $f_2^\bullet$ is a morphism of complexes. By uniqueness of the factorization, $f_1^\bullet$ is also a morphism of complex. Now we view $W^\bullet, V^{\bullet,B}, V^\bullet$ as objects in $D(E_\square)$. Since $V^{\bullet,B}$ is a nuclear complex, we know $f_1^\bullet$ is trace-class using [Bos23b, Lem. A.14] (here we use boundedness so that $W^\bullet$ is compact), thus $f^\bullet = f_2^\bullet \circ f_1^\bullet$ is trace-class by [CS22, Lem. 8.2].

For the second claim, we know $W^\bullet$ is a compact object by [And21, Lem. 5.40] in $D(E_\square)$ and $\mathrm{id}_{W^\bullet}$ is equivalent to a trace-class map. By [CS22, Prop. 9.4], $W^\bullet$ is a dualizable object in $D(E_\square)$ and by [And21, Cor. 5.51.1] it is isomorphic to a perfect complex. □

Proposition 2.33. *Assume $f^\bullet : W^\bullet \to V^\bullet$ is a morphism of bounded complexes of Banach E-vector spaces such that each f^i is a compact morphism and $f^\bullet$ induces a quasi-isomorphism. Then $W^\bullet$ (so is $V^\bullet$) is perfect as an object in $D(E_\square)$.*

Proof. Let $V_s^i = V^{-i,\vee}$ and $W_s^i = W^{-i,\vee}$ be Smith spaces. There is a morphism of bounded complexes of Smith spaces $f_s^\bullet : V_s^\bullet \to W_s^\bullet$ whose dual (taking $\underline{\mathrm{Hom}}_E(-, E)$ termwisely) is $f^\bullet$. By our assumption, f_s^i are trace-class. We claim that $f_s^\bullet$ induces a quasi-isomorphism. Then by Lemma 2.32, this implies $V_s^\bullet$ is a perfect complex thus $W^\bullet = R\underline{\mathrm{Hom}}(W_s^\bullet, E)$ is also a perfect complex.

To show $f_s^\bullet$ induces a quasi-isomorphism, let $C := \mathrm{Cone}(f_s^\bullet)$. Since $W^\bullet \simeq R\underline{\mathrm{Hom}}(W_s^\bullet, E)$, $V^\bullet \simeq R\underline{\mathrm{Hom}}(V_s^\bullet, E)$ and $\mathrm{Cone}(f) \simeq 0$, we get that $R\underline{\mathrm{Hom}}(C, E) \simeq 0$. Applying Lemma 2.25, we get $C \simeq 0$. This finished the proof. □

3. Equivariant vector bundles

In this section, we relate Bernstein–Zelevinsky duality of p-adic Lie groups with the Grothendieck–Serre duality of equivariant vector bundles (Theorem 3.16) and discuss equivariant vector bundles on the Drinfeld spaces in § 3.4.

Let G be a p-adic Lie group and let $K \subset G$ be an open compact subgroup. Let E be a finite extension of $\mathbb{Q}_p$.

3.1. **Bernstein–Zelevinsky duality of locally analytic representations.** We first recall the definition of the Bernstein–Zelevinsky duality of locally analytic representations. For $V \in \mathrm{Rep}^{la}_{E_\square}(G)$, its Bernstein–Zelevinsky dual ([SW25, Def. 5.4]) is given by:

$$\mathbb{D}_{\mathrm{BZ}}(V) := R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(V, \mathcal{C}^{la}_c(G,E)) \in D_\square(\mathcal{D}^{la}(G,E)). \tag{3.1}$$

Lemma 3.1. *Assume W is a bounded complex of locally analytic representations of K such that underlying modules of each W^i is a Smith space. Then $W^\vee \in \mathrm{Rep}^{la}_{E_\square}(K)$ where $W^\vee = R\underline{\mathrm{Hom}}_E(W,E)$ is endowed with dual action of K.*

Proof. Note that $R\underline{\mathrm{Hom}}_E(W_i,E) = \underline{\mathrm{Hom}}_E(W_i,E)$ using that Smith spaces are projective, the fact $\underline{\mathrm{Hom}}_E(V,-)(S) = \underline{\mathrm{Hom}}_E(V \otimes_{E_\square} E^\square[S],-))$ and [RJRC22, Prop. 3.12]. Hence $(W^\vee)_i = (W_{-i})^\vee$ is a Banach spaces. By [RJRC22, Prop. 4.41, Thm. 4.36(2)], since W_{-i} is a $\mathcal{D}^{la}(K,E)$-module, $(W^\vee)_i$ is a locally analytic representation. Hence $W^\vee \in \mathrm{Rep}^{la}_{E_\square}(K)$ since the latter is a full subcategory of $D_\square(\mathcal{D}^{la}(K,E))$ stable under small colimits [RJRC23, Prop. 3.3.3]. □

We write

$$\text{c-Ind}^G_K(-) = \mathcal{D}^{la}(G,E) \otimes^{L\square}_{\mathcal{D}^{la}(K,E)} - : D_\square(\mathcal{D}^{la}(K,E)) \to D_\square(\mathcal{D}^{la}(G,E))$$

for the compact induction functor.

Proposition 3.2 ([SW25, Thm. 5.3])**.** *Assume $W = W^\bullet$ is a bounded complex of locally analytic representations of K such that underlying modules of each W^i is a Smith space. Then we have a quasi-isomorphism in $D(\mathrm{Mod}_{E_\square}(D^{la}(G,E)))$*

$$\mathbb{D}_{\mathrm{BZ}}(\text{c-Ind}^G_K W) \simeq \text{c-Ind}^G_K W^\vee.$$

Proof. Let $\chi = \Delta_G^*|_K$ be the right character of Appendix A, and let ι be the involution $g \mapsto g^{-1}$ of G. We calculate that

$$\begin{aligned}
\mathbb{D}_{\mathrm{BZ}}(\text{c-Ind}^G_K W) &\simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(\text{c-Ind}^G_K W, \mathcal{C}^{la}_c(G,E)) \\
&\simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(K,E)}(W, \mathcal{C}^{la}_c(G,E)) \\
&\simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(K,E)}(E, R\underline{\mathrm{Hom}}_E(W, \mathcal{C}^{la}_c(G,E)) \\
&\simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(K,E)}(E, W^\vee \otimes^L_{E_\square} \mathcal{C}^{la}_c(G,E)) \\
&\simeq (\iota(W^\vee) \otimes^L_{E_\square} \chi) \otimes^L_{\mathcal{D}^{la}(K,E)} \mathcal{C}^{la}_c(G,E)[-\dim G] \\
&\simeq ((\iota(W^\vee) \otimes^L_{E_\square} \chi) \otimes^L_{\mathcal{D}^{la}(K,E)} \mathcal{C}^{la}(K,E)[-\dim G]) \otimes^L_{\mathcal{D}^{la}(K,E)} \mathcal{D}^{la}(G,E) \\
&\simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(K,E)}(E, W^\vee \otimes^L_{E_\square} \mathcal{C}^{la}(K,E)) \otimes^L_{\mathcal{D}^{la}(K,E)} \mathcal{D}^{la}(G,E) \\
&\simeq W^\vee \otimes^L_{\mathcal{D}^{la}(K,E)} \mathcal{D}^{la}(G,E) \simeq \text{c-Ind}^G_K W^\vee
\end{aligned}$$

where the second quasi-isomorphism is by adjunction; the third one follows from [RJRC23, Prop. A.2.1(4)]; the fourth one is because W is compact and $\mathcal{C}^{la}_c(G,E)$ is nuclear; the fifth and seventh isomorphisms use [RJRC23, Prop. 3.2.12]; the eighth is by Lemma 3.1 which tells that $W^\vee \simeq (W^\vee)^{Rla} = R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(K,E)}(E, W^\vee \otimes^L_{E_\square} \mathcal{C}^{la}(K,E))$ ([RJRC23, Def. 3.2.2]). □

3.2. **Locally analytic vector bundles.** In this subsection, we let G be a p-adic Lie group. Let E be a finite extension over $\mathbb{Q}_p$ and let X be an analytic adic space over E with a continuous action of G. We discuss the quotient analytic stack $[X/G^{la}]$ and vector bundles on it, viewing X as an analytic stack over $E_\square$. From now on, by an analytic stack we mean a Tate stack (Definition 2.15).

Let $H \subset G$ be a open compact subgroup. We also write H for the analytic adic space $\mathrm{Spa}(\mathcal{C}(H,E), \mathcal{C}(H,\mathcal{O}_E))$ and

$$H^{la} := \mathrm{AnSpec}((\mathcal{C}^{la}(H,E), \mathcal{O}_E)_\square).$$

Since $G = \sqcup_{g \in G/H} g.H$, we have a group stack $G^{la} = \sqcup_{g \in G/H} g.H^{la}$.

If G acts on an analytic adic space X over E, we say that the action of G on X is continuous if the (a priori set-theoretic) action map

$$\mathrm{act} : G \times X \to X \tag{3.2}$$

extends to an action of the adic space G on X, and the action is locally analytic if (3.2) is continuous and extends to an action

$$\mathrm{act}^{la} : G^{la} \times X \to X \tag{3.3}$$

where X is viewed as an analytic stack over $E_\square$.

Definition 3.3. Let X be an analytic adic space with a locally analytic action of G. We define an analytic stack

$$[X/G^{la}] := \mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}}(G^{la})^n \times X,$$

the geometric realization of the simplicial object $((G^{la})^n \times X)_{[n]\in\Delta^{\mathrm{op}}}$ described in [RJRC23, Def. 4.3.1].

By the construction of the six functor formalism, we have (see discussions after [HM24, Exam. 3.4.5])

$$D([X/G^{la}]) = \lim_{[n]\in\Delta} D((G^{la})^n \times X). \tag{3.4}$$

If $X = *$, then by [RJRC23, Thm. 4.3.3]:

$$D([*/G^{la}]) \simeq \mathrm{Rep}^{la}_{E_\square}(G). \tag{3.5}$$

If $X = \mathrm{AnSpec}\, A$ where A is a solid analytic affinoid ring (Definition 2.5) and G is compact, then by [Mik26a, Prop. 2.47] (taking $h = \infty$ in *loc. cit.*)

$$D([X/G^{la}]) \simeq \mathrm{Rep}^{la}_{A}(G)$$

where the latter is the category of semilinear A-modules in [Mik26a, Def. 2.46].

We will focus on classical examples of equivariant vector bundles.

Lemma 3.4. *Let X be a rigid analytic variety over E equipped with an action of G. Suppose that for any affinoid open $U \subset X$, there exists an open compact subgroup $K_U \subset G$ which stabilizes U and the induced action $K_U \times \mathcal{O}(U) \to \mathcal{O}(U)$ is continuous (this is the definition of continuous action in* [Sch18, Def. 2.1, Lem. 2.2]*). Then the G-action on X extends to an action of G or G^{la} on X as adic spaces or analytic stacks respectively.*

Proof. To extend the action $G^{(la)} \times X \to X$, it suffices to work locally and define actions $G^{(la)} \times U \to X$ for $U \subset X$ affinoid. Since U is stabilized by a compact open subgroup of G, we may easily reduce to the case when G is a compact group and $X = \mathrm{Spa}(A, A^\circ)$ is an affinoid space. Since the action of G on X is continuous, the orbit map $A \to \mathrm{Hom}(G, A), v \mapsto (g \mapsto g.v)$ has image in $\mathcal{C}(G, A)$ the continuous functions from G to A, which gives a map $A \to \mathcal{C}(G, A) \simeq \mathcal{C}(G, E) \otimes_{E_\square} A$ (use [Eme17, (2.1.3)] and [RJRC22, Lem. 3.13]). This makes A a comodule over the Hopf algebra $\mathcal{C}(G, E)$ (cf. [RJRC23, Lem. 6.1.3]). Taking the adic spectra we get the map $G \times X \to X$. To get the locally analytic statement, by [RJRC23, Lem. 4.3.4], it suffices to show that G acts locally analytically on A. This follows from Lemma 3.5 below. □

Lemma 3.5. *Let A be an affinoid algebra over E, and let G be a p-adic Lie group acting continuously on A by E-algebra automorphisms. Let M be a finite A-module equipped with a continuous A-semilinear action of G (with respect to the canonical Banach topology on M). Then the actions of G on A and on M are locally analytic.*

Proof. Choose a surjection $E\langle T_1, \cdots, T_n\rangle \twoheadrightarrow A$ and let A_0 be the image of $\mathcal{O}_E\langle T_1, \cdots, T_n\rangle$ which is an open lattice of A. We may suppose that G is compact and after shrinking G, we may assume that A_0 is stabilized by G (by continuity). The group G acts continuously, hence smoothly, on the discrete space A_0/ϖ_E which is finitely generated over $\mathcal{O}_E/\varpi_E$. Thus there exists an open subgroup $H \subset G$ which fixes a set of generators of A_0/ϖ_E. Consequently, as G acts by algebra endomorphisms, H acts trivially on A_0/ϖ_E. Applying [RJRC23, Prop. 3.4.1], the G-action on A is locally analytic. The proof for M is similar, cf. [RC26a, Exam. 2.3.8]. □

Next, suppose that X is an analytic adic space with a continuous G-action $\mathrm{act} : G \times X \to X$. Let $\mathcal{F}$ be a G-equivariant vector bundle on X. This means a locally free $\mathcal{O}_X$-module with an isomorphism $\mathrm{act}^*\mathcal{F} \simeq \pi^*\mathcal{F}$ satisfying the usual cocycle condition, where $\pi : G \times X \to X$ is the natural projection. Similarly, we say the G-equivariant vector bundle $\mathcal{F}$ is locally analytic if the G action induces an isomorphism $(\mathrm{act}^{la})^*\mathcal{F} \simeq (\pi^{la})^*\mathcal{F}$ where $\pi^{la} : G^{la} \times X \to X$ is the natural projection.

Lemma 3.6. *Let X be a rigid analytic variety over E equipped with a continuous action of G satisfying the assumption in Lemma 3.4. Let $\mathcal{F}$ be a G-equivariant vector bundle on X. Then $\mathcal{F}$ is locally analytic and corresponds to an object in $D([X/G^{la}])$.*

Proof. The vector bundle $\mathcal{F}$ is locally analytic by Lemma 3.5. Assume that an open affinoid $U = \mathrm{Spa}(A, A^+) \subset X$ is stabilized by an open compact subgroup K of G. The space $\mathcal{C}^{la}(K, E)$ is a filtered colimit of Banach spaces, hence is flat over $E_\square$ [Bos23a, Cor. A.65]. Thus $(K^{la})^n \times U = \mathrm{AnSpec}((A, A^+)_\square \otimes_{E_\square} \mathcal{C}^{la}(K^n, E))$. It's then formal to check that $\mathcal{F}(U)$ defines an object in $D(U/K^{la})$ and $\mathcal{F}$ defines an object in $D(X/G^{la})$. □

Lemma 3.7. *Let X be an analytic adic space equipped with a locally analytic action of G. The projection $p_X : X \to [X/G^{la}]$ is a D-cover (Definition 2.13). In particular, p_X satisfies universal $*$-descent and universal $!$-descent.*

Proof. By [RC24, Prop. 6.2.3], the map $* \to [*/G^{la}]$ is a D-cover. The result follows since universal $*$-descent and $!$-descent are stable under pullback by definition and p_X is the pullback of $* \to [*/G^{la}]$ along $[X/G^{la}] \to [*/G^{la}]$. □

Lemma 3.8. *Let $f\colon X \to Y$ be a G^{la}-equivariant morphism of analytic adic spaces equipped with locally analytic actions of G. The square*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle p_Y} \\ [X/G^{la}] & \xrightarrow[\bar{f}]{} & [Y/G^{la}] \end{array}$$

is cartesian. If f is $!$-able (i.e. in the class E'' after Definition 2.15), then $\bar{f}$ is $!$-able, and $p_Y^* \bar{f}_! \simeq f_! p_X^*$ as functors $D([X/G^{la}]) \to D(Y)$. Moreover, the natural morphism $p_Y^* \bar{f}_* \to f_* p_X^*$ is an equivalence of functors $D([X/G^{la}]) \to D(Y)$ if f is quasi-compact and separated.

Proof. Both p_X and p_Y are pullbacks of $* \to [*/G^{la}]$, hence the square is cartesian. By Lemma 3.7, p_Y satisfies universal $!$-descent. Since to check if $h_!$ is defined for a morphism h, one can check $!$-locally on the target, we see $\bar{f}$ is $!$-able if f is [HM24, Thm. 3.4.11]. The equivalence $p_Y^* \bar{f}_! \simeq f_! p_X^*$ is the proper base change [HM24, Prop. 3.1.8] since p_Y is a D-cover.

We remain to prove the last statement. The natural morphism $p_Y^* \bar{f}_* \to f_* p_X^*$ arises as usual from the adjoint of $f^* p_Y^* \bar{f}_* \simeq p_X^* \bar{f}^* \bar{f}_* \to p_X^*$. For any map $\alpha : [n] \to [m]$ in Δ, write ψ_α for the corresponding maps of the simplicial objects, and consider the following Cartesian diagram:

$$\begin{array}{ccc} (G^{la})^m \times X & \xrightarrow[f_m]{} & (G^{la})^m \times Y \\ \downarrow{\scriptstyle \psi_{X,\alpha}} & & \downarrow{\scriptstyle \psi_{Y,\alpha}} \\ (G^{la})^n \times X & \xrightarrow[f_n]{} & (G^{la})^n \times Y. \end{array} \tag{3.6}$$

We claim that we have equivalences

$$f_{m*} \psi_{X,\alpha}^* \simeq \psi_{Y,\alpha}^* f_{n*} \tag{3.7}$$

This equivalence holds when G is compact and both X and Y are affinoid by the proof of [Mik26a, Cor. 2.42]. We prove our case by reducing to the affinoid cases. We may assume α is a degeneracy map or a face map.

We prove the equivalence (3.7) in the case when $\psi_{X,\alpha}$ is the action map $G^{la} \times X \to X$; the other cases are similar or easier. Take an affinoid cover $(U_i)_i$ with open compact stabilizers K_i of U_i. Write $V_i = f_0^{-1}(U_i)$. Consider the map $\sqcup_{g \in G/K_i} K_i g^{-1} \times g.V_i \to \sqcup_{g \in G/K_i} K_i g^{-1} \times g.U_i$. We note that open immersions are D-étale by the discussion in Example 2.16, hence $*$-pushforwards commute with $*$-pullback along open immersions by [HM24, Lem. 4.5.13]. Since $\sqcup_{g \in G/K_i} K_i g^{-1} \times g.U_i$ is an open covering of the preimage of U_i in $G^{la} \times Y$, and f_{1*} commutes with direct product, we can check (3.7) after pulling back (3.6) to each $K_i g^{-1} \times g.U_i$. For this we can assume that G is compact and Y is affinoid. Let $i : V \hookrightarrow X$ be an open immersion of an affinoid subspace and we may assume that V is stabilized by an open subgroup $K \subset G$. We can cover $G \times X$ by $Kg^{-1} \times g.V, g \in G/K$. By the affinoid case in [Mik26a, Cor. 2.42], we have $(f_1|_{Kg^{-1} \times g.V})_* (\psi_{X,\alpha}|_{Kg^{-1} \times g.V})^* \simeq (\psi_{Y,\alpha}|_{Kg^{-1} \times Y})^* (f_0|_V)_*$. Since f is quasi-compact and separated, we may take a finite affinoid open cover $(U_i)_{i=1,\cdots,l}$ of X and let K be an open subgroup of G stabilizing all U_i, as well as their affinoid intersections $U_J = \cap_{i \in J} U_i$ for any $J \subset \{1, \cdots, l\}$. Write $i_J : U_J \to X$ for the open immersion. Since an open covering is a D-cover, by descent, $\mathcal{F}$ is isomorphic to the total complex of the alternating Čech complex $[\prod_{1 \le j \le l} i_{\{j\}*} \mathcal{F}|_{U_j} \to \prod_{|J|=i} i_{J*} \mathcal{F}|_{U_J} \to \cdots \to j_{\{1,\cdots,l\}*} \mathcal{F}|_{U_1 \cap \cdots \cap U_l}]$

which is a finite limit of the objects $i_{J*}\mathcal{F}|_{U_J}$. Since both Ψ_α^* and f_* commute with finite limits (as $D(X)$ is stable), the equivalence (3.7) follows.

Hence the diagram obtained after applying $D^*(-)$ to (3.6) (with $*$-pullbacks) is right adjointable in the sense of [Lur17, Def. 4.7.4.13], with right adjoints the $*$-pushforwards. Since $\bar{f}_*$ is by definition of the right adjoint of $\bar{f}^* = \lim_{[n]} f_n^*$ (3.4) and right adjoints commute with limits, it is formal to verify that $\bar{f}_* = \lim_{[n]} f_{n*}$ (cf. [Lur17, Cor. 4.7.4.18]). The result follows. □

Remark 3.9. The last statement of Lemma 3.8 is wrong without the quasi-compact assumption. For example if we take $Y = *$ and $X = \Omega_{\mathrm{Dr},E}$ a Stein space and take $\mathcal{F}$ a G-equivariant vector bundle on $\Omega_{\mathrm{Dr},E}$, then $H^0(\Omega_{\mathrm{Dr},E}, \mathcal{F})$ is in general not locally analytic, but the linear dual of a locally analytic representation of compact type. Hence $f_*\mathcal{F}$ does not lie in the image of $D([*/G^{la}]) \to D(*)$ via $*$-pullback.

3.3. **Two resolutions of Fargues.** Let X be a smooth partially proper rigid analytic variety over E (countable at infinity) of pure dimension d. Suppose G acts continuously on X as in Lemma 3.4. Let $\mathcal{F}$ be an equivariant vector bundle on X defining an object in $D([X/G^{la}])$ by Lemma 3.6. We follow the strategy of Fargues in [Far] to calculate the Bernstein–Zelevinsky dual of $R\Gamma_c(X, \mathcal{F}) \in \mathrm{Rep}_{E_\square}^{la}(G)$ using Čech complexes.

We take $U \subset X$ be an affinoid open subset and we assume the following assumption:

Assumption 3.10. We assume that the affinoid open $U \subseteq X$ is stablized by a compact open $K \subseteq G$ such that $GU = X$ and the set $\{g \in G/K : U \cap g.U \neq \emptyset\}$ is a finite set.

For a compact open subgroup $K \subseteq G$, we get a map $q : [*/K^{la}] \to [*/G^{la}]$. It is not difficult to see that q^* corresponds, under the equivalence (3.5), to the restriction functor. So q_* corresponds to the locally analytic induction functor $\mathrm{Ind}_K^G(-)$ and $q_!$ corresponds to locally analytic compact induction functor c-Ind_K^G (see discussions after [RJRC23, Corollary 3.3.7], cf. [HM24, § 1.4]). We have a commutative diagram of Tate stacks

$$\begin{array}{ccccc} [U/K^{la}] & \xrightarrow{j} & [X/K^{la}] & \xrightarrow{i} & [X/G^{la}] \\ {\scriptstyle p_U}\downarrow & & & & \downarrow{\scriptstyle p_X} \\ [*/K^{la}] & = & [*/K^{la}] & \xrightarrow[q]{} & [*/G^{la}] \end{array} \tag{3.8}$$

All morphisms in the above diagram are !-able, since !-ability is !-local on the target and we have Lemma 3.7.

Lemma 3.11. *The map $f : [U/K^{la}] \to [X/G^{la}]$ satisfies universal D^*-descent and universal $D^!$-descent*

Proof. We apply [HM24, Lem. 4.7.1]. We need to check f is D-suave [HM24, Def. 4.5.1] and f^* is conservative. By Lemma 3.7, the map $X \to [X/G^{la}]$ is a D-cover. In particular, it satisfies universal $*$-descent and universal !-descent and is conservative.

By our assumption, the map $G \times^K U = \sqcup_{g \in G/K} g.U \simeq [U/K^{la}] \times_{[X/G^{la}]} X \to X$ is an open cover. Thus it is D-suave (even D-étale) and conservative by discussions in Example 2.16. Now the claim follows because the property for a map to be D-suave can be checked after pulling back to a universal D^*-cover on the target ([HM24, Lemma 4.5.7]). □

Let $\{[U/K^{la}]^{[n]}\}_{[n] \in \Delta^{\mathrm{op}}}$ be the Čech nerve of f. For $\alpha = (1, \overline{g_1}, \cdots, \overline{g_n}) \in G\backslash(G/K)^{n+1}$, we put $U_\alpha := U \cap g_1.U \cdots \cap g_n.U$, $K_\alpha := K \cap g_1 K g_1^{-1} \cdots \cap g_n K g_n^{-1}$. Note that by our assumption of the covering $\{g.U\}_{g \in G/K}$, for each $n \in \mathbb{N}$, the set $\{\alpha \in G\backslash(G/K)^{n+1} : U_\alpha \neq \emptyset\}$ is a finite set.

Lemma 3.12. *We have an equivalence of Tate stacks*

$$\begin{aligned} [U/K^{la}]^{[n]} &= [U/K^{la}] \times_{[X/G^{la}]} \times \cdots \times_{[X/G^{la}]} [U/K^{la}] \\ &\simeq \sqcup_{\alpha \in G\backslash(G/K)^{n+1}} [U_\alpha / K_\alpha^{la}], \end{aligned}$$

where U_α, K_α are as above.

Proof. Pulling back along $X \to [X/G^{la}]$ it amounts to show

$$(G \times^K U)^{n/X} \simeq \sqcup_{\alpha \in G\backslash(G/K)^{n+1}} G \times^{K_\alpha} U_\alpha$$

which is formal. □

Following the language of Fargues, we define functors:

$$\mathrm{Res}^{X/G}_{U/K} := j^*i^* : D([X/G^{la}]) \to D([U/K^{la}]),$$
$$\text{c-Ind}^{X/G}_{U/K} := i_!j_! : D([U/K^{la}]) \to D([X/G^{la}]).$$

By !-descent, Lemma 3.11 and Lemma 3.12, we get a resolution

$$\mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}} C^n(\mathcal{F}) \xrightarrow{\sim} \mathcal{F} \tag{3.9}$$

of $\mathcal{F}$ in $D([X/G^{la}])$, where

$$C^n(\mathcal{F}) := \bigoplus_{\alpha\in G\backslash(G/K)^{n+1}} \text{c-Ind}^{X/G}_{U_\alpha/K_\alpha}\mathrm{Res}^{X/G}_{U_\alpha/K_\alpha}\mathcal{F}.$$

We can also view $\mathcal{F}$ as a solid abelian sheaf, and consider the classical compactly supported cohomology as in § 2.3. For $\mathcal{F}_\alpha$ a K_α-equivariant vector bundle on U_α, let

$$\mathrm{Ind}^{X/G}_{U_\alpha/K_\alpha}\mathcal{F}_\alpha := \prod_{g\in G/K_\alpha} g^*j_{U_\alpha *}\mathcal{F}_\alpha \simeq \bigoplus g^*j_{U_\alpha *}\mathcal{F}_\alpha$$

be a G-equivariant solid abelian sheaf on X (note that $j_{U_\alpha *}\mathcal{F}_\alpha = Rj_{U_\alpha *}\mathcal{F}_\alpha$ as in the proof of Proposition 2.21). There is a resolution

$$\mathcal{F} \to \lim_{[n]\in\Delta} D^n(\mathcal{F}) \tag{3.10}$$

of $\mathcal{F}$ given by

$$D^n(\mathcal{F}) = \bigoplus_{\alpha\in G\backslash(G/K)^{n+1}} \mathrm{Ind}^{X/G}_{U_\alpha/K_\alpha}\mathcal{F}|_{U_\alpha}.$$

In other words, $\mathcal{F}$ is quasi-isomorphic to the cohomological complex (see [Wei94, Thm. 8.3.8])

$$D^\bullet(\mathcal{F}) := [D^0(\mathcal{F}) \to D^1(\mathcal{F}) \to \cdots \to D^n(\mathcal{F}) \to \cdots] \tag{3.11}$$

One can check the above complex is a resolution by restricting to U using G-translations, together with the very standard statements [Sta25, Tag 02FU].

Remark 3.13. The differentials of the complex $D^\bullet(\mathcal{F})$ is given using the following: It is easy to check that

$$\mathrm{Hom}_{K_\alpha}(\mathcal{F}|_{U_\alpha}, \mathcal{F}_\alpha) \cong \mathrm{Hom}_G(\mathcal{F}, \mathrm{Ind}^{X/G}_{U_\alpha/K_\alpha}\mathcal{F}_\alpha)$$

where the Hom sets are taken in the category of equivariant solid abelian sheaves. Let $\alpha = (1, \overline{g_1}, \cdots, \overline{g_n}) \in G\backslash(G/K)^{n+1}$ and $\beta = (\alpha, \overline{g_{n+1}}) \in G\backslash(G/K)^{n+2}$, then we have maps

$$\begin{aligned}\mathrm{Hom}_G(\mathrm{Ind}^{X/G}_{U_\alpha/K_\alpha}\mathcal{F}|_{U_\alpha}, \mathrm{Ind}^{X/G}_{U_\beta/K_\beta}\mathcal{F}|_{U_\beta}) &= \mathrm{Hom}_G(\mathrm{Ind}^{X/G}_{U_\alpha/K_\alpha}\mathcal{F}|_{U_\alpha}, \mathrm{Ind}^{X/G}_{U_\alpha/K_\alpha}\mathrm{Ind}^{U_\alpha/K_\alpha}_{U_\beta/K_\beta}\mathcal{F}|_{U_\beta}) \\ &\leftarrow \mathrm{Hom}_{K_\alpha}(\mathcal{F}|_{U_\alpha}, \mathrm{Ind}^{U_\alpha/K_\alpha}_{U_\beta/K_\beta}\mathcal{F}|_{U_\beta}) \\ &= \mathrm{Hom}_{K_\beta}(\mathcal{F}|_{U_\beta}, \mathcal{F}|_{U_\beta}).\end{aligned}$$

Lemma 3.14. *We have natural isomorphisms*

(1) $R\Gamma_c(X, \text{c-Ind}^{X/G}_{U/K}\mathcal{F}) \simeq \text{c-Ind}^G_K R\Gamma_c(U, \mathcal{F}) \simeq \text{c-Ind}^G_K H^d_c(U, \mathcal{F})[-d]$ *and*

(2) $R\Gamma^{cl}_c(X, \mathrm{Ind}^{X/G}_{U/K}\mathcal{F}|_U) \simeq \text{c-Ind}^G_K R\Gamma(U, \mathcal{F}|_U) \simeq \text{c-Ind}^G_K H^0(U, \mathcal{F})$

in $D_\square(\mathcal{D}^{la}(G, E))$.

Proof. (1) By Lemma 3.8 (proper base change) and (3.8), $R\Gamma_c(X, \mathcal{F}) \in \mathrm{Rep}^{la}_{E_\square}(G)$ corresponds to $p_{X!}\mathcal{F}$, we have $R\Gamma_c(X, \text{c-Ind}^{X/G}_{U/K}\mathcal{F}) \simeq R\Gamma_c(X, i_!j_!\mathcal{F}) \simeq p_{X!}i_!j_!\mathcal{F} \simeq q_!p_{U!}\mathcal{F} \simeq \text{c-Ind}^G_K R\Gamma_c(U, \mathcal{F})$. The concentration of $R\Gamma_c(U, \mathcal{F})$ in degree d is Proposition 2.28.

(2) By definition $\mathrm{Ind}^{X/G}_{U/K}\mathcal{F}|_U = \prod_{g\in G/K} g^*j_{U*}j_U^*\mathcal{F}$. By Proposition 2.21, the underlying (complex of) module of $R\Gamma^{cl}_c(X, \mathrm{Ind}^{X/G}_{U/K}\mathcal{F}|_U)$ is $\oplus_{g\in G/K} R\Gamma(U, \mathcal{F})$. By Tate's acyclicity, $R\Gamma(U, \mathcal{F})$ concentrates in degree 0 and by Lemma 3.4, the K-action on it is locally analytic. Since both sides concentrate in a single degree, the isomorphism is $\mathcal{D}^{la}(G, E)$-equivalent. □

Proposition 3.15. *We have natural quasi-isomorphisms of complexes in* $\mathrm{Rep}^{la}_{E_\square}(G)$*:*

(1) (!-resolution)

$$\mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}}(\oplus_{\alpha\in G\backslash(G/K)^{n+1}} \text{c-Ind}^G_{K_\alpha} H^d_c(U_\alpha, \mathcal{F})[-d]) \simeq R\Gamma_c(X, \mathcal{F}); \tag{3.12}$$

(2) (∗-resolution)

$$R\Gamma_c(X,\mathcal{F}) \simeq \lim_{[n]\in\Delta}(\oplus_{\alpha\in G\backslash(G/K)^{n+1}}\text{c-Ind}_{K_\alpha}^G H^0(U_\alpha,\mathcal{F})). \tag{3.13}$$

Proof. (1) The result follows by taking $R\Gamma_c(X,-)$ on the resolution (3.9) which commutes colimits and applying Lemma 3.14(1).

(2) Applying $R\Gamma_c^{cl}(X,-)$ to the resolution (3.10), since each term of (3.10) is acyclic for the functor by Lemma 3.14(2), we see

$$R\Gamma_c^{cl}(X,\mathcal{F}) \simeq [\text{c-Ind}_K^G H^0(U,\mathcal{F}) \to \oplus_{\alpha=(1,\overline{g}_1)\in G\backslash(G/K)^2}\text{c-Ind}_{K_\alpha}^G H^0(U_\alpha,\mathcal{F}) \to \cdots].$$

We conclude by applying Corollary 2.24. □

Theorem 3.16. *Let $\mathcal{F}$ be a G-equivariant vector bundle on X which is partially proper and countable at infinity. Suppose that there exists an affinoid open subspace $U \subset X$ satisfying Assumption 3.10. Then we have a quasi-isomorphism of $\mathcal{D}^{la}(G,E)$-modules*

$$R\Gamma_c(X,\mathcal{F}^\vee\otimes\omega_X) \simeq \mathbb{D}_{\mathrm{BZ}}(R\Gamma_c(X,\mathcal{F})),$$

where $\mathcal{F}^\vee$ is the dual vector bundle and ω_X is the dualizing complex on X which is a line bundle concentrating in degree $-d$.

Proof. Taking the Bernstein–Zelevinsky dual (3.1) on (3.12), we get following quasi-isomorphisms of $\mathcal{D}^{la}(G,E)$-representations:

$$\begin{aligned}
\mathbb{D}_{\mathrm{BZ}}(R\Gamma_c(X,\mathcal{F})) &\simeq \lim_{[n]\in\Delta}(\oplus_{\alpha\in G\backslash(G/K)^{n+1}}\mathbb{D}_{\mathrm{BZ}}(\text{c-Ind}_{K_\alpha}^G R\Gamma_c(U_\alpha,\mathcal{F})))\\
&\simeq \lim_{[n]\in\Delta}(\oplus_{\alpha\in G\backslash(G/K)^{n+1}}\text{c-Ind}_{K_\alpha}^G R\Gamma_c(U_\alpha,\mathcal{F})^\vee)\\
&\simeq \lim_{[n]\in\Delta}(\oplus_{\alpha\in G\backslash(G/K)^{n+1}}\text{c-Ind}_{K_\alpha}^G R\Gamma(U_\alpha,\mathcal{F}^\vee\otimes\omega_X))\\
&\simeq R\Gamma_c(X,\mathcal{F}^\vee\otimes\omega_X)
\end{aligned}$$

where the first quasi-isomorphism is because the direct sum in each degree is essentially finite, the second one comes from Proposition 3.2, using that $H_c^d(U_\alpha,\mathcal{F})$ is a locally analytic K-representation whose underlying module is a Smith space (Proposition 2.28), the third one is Serre duality (Theorem 2.18) and the last one is (3.13). □

Remark 3.17. One can obtain Theorem 3.16 without using classical compactly supported cohomology. In other words, (3.13) can be established using the 6-functor formalism as follows. By $*$-descent and Assumption 3.10, a quasi-isomorphism $\mathcal{F} \simeq \lim_{[n]\in\Delta} D^n(\mathcal{F})$ as in (3.11) exists in $D(X)$, where $D^n(\mathcal{F})$ is given by $\oplus_{\alpha\in G\backslash(G/K)^{n+1}}\mathrm{Ind}_{U_\alpha/K_\alpha}^{X/G}\mathcal{F}|_{U_\alpha} \simeq \oplus_{\beta\in(G/K)^{n+1}} j_{\beta*}\mathcal{F}|_{U_\beta}$ (for $\beta=(\overline{g}_1,\overline{g}_1,\cdots,\overline{g}_n)$, $U_\beta=\cap_{i=0}^n g_i.U$ and $j_\beta: U_\beta\hookrightarrow X$). Replacing the Čech complex by the alternating one $D^{\mathrm{alt},\bullet}(\mathcal{F})$ [Sta25, Tag 01FG] (cf. the proof of Proposition 4.7), we may assume that $D^{\mathrm{alt},n}(\mathcal{F})=0$ for n large enough. Since $R\Gamma_c(X,-)$ preserves finite limits of the stable ∞-categories, we see $R\Gamma_c(X,\mathcal{F}) \simeq \varprojlim_{[n]\in\Delta} R\Gamma_c(X,D^{\mathrm{alt},n}(\mathcal{F}))$. On the other hand, for each affinoid open $V=\mathrm{Spa}(A,A^\circ)\hookrightarrow X$ consider the factorization $V \xrightarrow{j_V} \overline{V}=\mathrm{Spa}(A,\mathbb{Z}_p) \xrightarrow{\overline{i}_V} X \xrightarrow{p_X} *$ (Remark 2.12). It is easy to see that $\overline{i}_{V,*}j_{V*}j_V^*\overline{i}_V^*\mathcal{F} \simeq \overline{i}_{V,*}\overline{i}_V^*\mathcal{F}$ using that $\mathcal{F}$ is a vector bundle. We can factor $\overline{i}$ as $\overline{V}\xrightarrow{\Gamma_{\overline{i}}}\overline{V}\times X\xrightarrow{\mathrm{pr}}X$. Since $\Gamma_{\overline{i}*}=\Gamma_{\overline{i}!}$, $\mathrm{pr}_*=\mathrm{pr}_!$, we have $\overline{i}_{V!}=\overline{i}_{V*}$. Hence $R\Gamma_c(X,(\overline{i}_V j_V)_*\mathcal{F}|_V) = p_{X!}\overline{i}_{V,*}j_{V*}j_V^*\overline{i}_V^*\mathcal{F} \simeq p_{X!}\overline{i}_{V,!}\overline{i}_V^*\mathcal{F}$. This can be used to replace Lemma 3.14(2).

3.4. **Application to the Drinfeld spaces.** We apply Theorem 3.16 to the Drinfeld spaces and discuss examples of equivariant vector bundles.

Let L be a finite extension of $\mathbb{Q}_p$ with a uniformizer ϖ_L and $E\supset L$. Let $n\geq 2$, we let $X=\Omega_{\mathrm{Dr},E}=\mathbb{P}_E^{n-1,\mathrm{an}}\setminus\cup_{H\in\mathcal{H}}H$ be the Drinfeld space over E of dimension $n-1$ (1.1). The group $G:=\mathrm{PGL}_n(L)$ acts on $X\subset\mathbb{P}_E^{n-1,\mathrm{an}}$ naturally and continuously.

Lemma 3.18. *There exists a sequence of affinoid open subspaces $U_{r_1}\subset U_{r_2}\subset\cdots U_{r_k}\subset\cdots, k\in\mathbb{N}$ of $\Omega_{\mathrm{Dr},E}$ such that $\Omega_{\mathrm{Dr},E}=G.U_{r_k}$ for all $k\geq 1$ and satisfying the following:*

(1) There exists an open compact subgroup $K\subset G$ stabilizing U_{r_k} and there exists a finite subset $\mathcal{S}\subset G/K$ such that the set $\{g\in G/K \mid U_{r_k}\cap gU_{r_k}\neq\emptyset\}$ is contained in $\mathcal{S}$ for all k.

(2) For any $k_1 < k_2$, $U_{r_{k_1}}$ is relatively compact in $U_{r_{k_2}}$ in the sense of [BGR84, § 9.6.2].

Proof. We can assume $E = L$. There is a G-equivariant map

$$\lambda : |\Omega_{\mathrm{Dr},E}| \to |\mathcal{BT}| \tag{3.14}$$

by [Dri74, § 6, A)]. Here $|\mathcal{BT}|$ can be identified with the set of norms on L^n up to similarity, and λ sends $(z_0, \cdots, z_{n-1}) \in \Omega_{\mathrm{Dr},E} \subset \mathbb{P}_L^{n-1}$ to the norm $(a_0, \cdots, a_{n-1}) \in L^n \mapsto |\sum_i z_i a_i|$ (with $|p| = p^{-1}$). There is a metric ρ on $|\mathcal{BT}|$ such that $\rho(x, y) \leq 1$ for any $x, y \in \sigma$ for any simplex σ of $\mathcal{BT}$. Take a vertex v_0 (0-simplex) on $\mathcal{BT}$ and for $r \in \mathbb{Q}$, set $V_r = \{a \in |\mathcal{BT}| \mid \rho(a, v_0) \leq r\} \subset |\mathcal{BT}|$ and

$$U_r := \{z \in \Omega_{\mathrm{Dr},E} \mid \rho(v_0, \lambda(z)) \leq r\} = \lambda^{-1}(V_r).$$

By [Dri74, Prop. 6.1], U_r is an affinoid open subspace of $\Omega_{\mathrm{Dr},E}$ and if $r < r'$, then U_r is relatively compact in $U_{r'}$. If $r > 1$, then for any simplex $\sigma \in \mathcal{BT}$, there exists $g \in G$ such that $g.\sigma \in V_r \subset V_2$ since G acts transitively on the vertices of $\mathcal{BT}$. Thus $G.V_r = |\mathcal{BT}|$ and consequently $G.U_r = \Omega_{\mathrm{Dr},E}$. Take an open compact subgroup K fixing v_0 (and we may assume $K = \mathrm{PGL}_n(\mathcal{O}_L)$). Since the G-action is isometric, K stabilizes V_r for all r. If $\rho(g.v_0, v_0) > 2r$, then $g(V_r) \cap V_r = \emptyset$. Take $\mathcal{S} = \{g \in G/K \mid \rho(g.v_0, v_0) \leq 4\}$. Then $\mathcal{S}$ is a finite set, and $g(V_r) \cap V_r = \emptyset$ for all $g \notin \mathcal{S}$ and $r < 2$. Consequently $g(U_r) \cap U_r = \emptyset$ for all $g \notin \mathcal{S}$ and $r < 2$. Now taking a sequence $r_1 < r_2 < \cdots < r_k < \cdots$ in $\mathbb{Q} \cap (1, 2)$, we get the required sequence of affinoid subspaces U_{r_k}. □

Corollary 3.19. *Let $\mathcal{F}$ be a G-equivariant vector bundle on $\Omega_{\mathrm{Dr},E}$. Then there is a quasi-isomorphism of $\mathcal{D}^{la}(G, E)$-modules*

$$R\Gamma_c(\Omega_{\mathrm{Dr},E}, \mathbb{D}_{\mathrm{GS}}(\mathcal{F})) \simeq \mathbb{D}_{\mathrm{BZ}}(R\Gamma_c(\Omega_{\mathrm{Dr},E}, \mathcal{F})).$$

Proof. By Lemma 3.18, Assumption 3.10 is satisfied for $\Omega_{\mathrm{Dr},E}$. we can apply Theorem 3.16. □

We now give examples of equivariant vector bundles on $\Omega_{\mathrm{Dr},E}$. Let ϖ_L be a uniformizer of L. Let $\breve{L}$ be the completion of the maximal unramified extension of L. Let D_L be the division algebra over L of invariant $1/n \in \mathbb{Q}/\mathbb{Z}$ with the maximal order $\mathcal{O}_{D_L}$. Let $D_L^\times, \mathcal{O}_{D_L}^\times$ be the group of units. Consider the Drinfeld tower

$$\cdots \to \breve{\mathcal{M}}_m \to \cdots \to \breve{\mathcal{M}}_1 \to \breve{\mathcal{M}}_0 \to \Omega_{\mathrm{Dr},\breve{L}} \tag{3.15}$$

where each $\breve{\mathcal{M}}_m$ is a rigid analytic space over $\breve{L}$ and is a finite étale $(\mathcal{O}_{D_L}/\varpi_L^m)^\times$-torsor over $\breve{\mathcal{M}}_0$. Theses spaces are essentially considered in [Dri76, § 3], except that in the moduli description of $\breve{\mathcal{M}}_m$, we do not require the quasi-isogeny of special formal $\mathcal{O}_{D_L}$-modules is of height 0. By [RZ96, Thm. 3.72], there is an isomorphism $\breve{\mathcal{M}}_0 \simeq \Omega_{\mathrm{Dr},\breve{L}} \times \mathbb{Z}$ compatible with the Weil descent data described in *loc. cit.*.

The tower $\{\breve{\mathcal{M}}_m\}_m$ admits a continuous action of $(D_L^\times \times \mathrm{GL}_n(L))/L^\times$ where $L^\times \subset D_L^\times \times \mathrm{GL}_n(L) : g \mapsto (g, g^{-1})$ is in the center. The period map $\{\breve{\mathcal{M}}_m\}_m \to \Omega_{\mathrm{Dr},\breve{L}}$ is $D_L^\times \times \mathrm{GL}_n(L)$-equivariant for the natural action of $\mathrm{GL}_n(L)$ and trivial action of $D_L^\times$-action on $\Omega_{\mathrm{Dr},\breve{L}}$ exhibiting the Drinfeld tower as a pro-étale $D_L^\times$-torsor over $\Omega_{\mathrm{Dr},\breve{L}}$.

The Weil descent datum on $\breve{\mathcal{M}}_m$ is not effective, but that on $\varpi_L^{\mathbb{Z}}\backslash\breve{\mathcal{M}}_m$ is [RZ96, Thm. 3.49]. Denote its descent to L by Σ_m and put $\Sigma_{m,E} := \Sigma_m \otimes_L E$. See [DLB17, §1.1 and §3.1] for the $\mathrm{GL}_2(\mathbb{Q}_p)$-case. We write $\overline{G} := \mathrm{GL}_n(L)/\varpi_L^{\mathbb{Z}}$ and $\overline{D}_L^\times := D_L^\times/\varpi_L^{\mathbb{Z}}$. Then the tower

$$\cdots \to \Sigma_{m,E} \to \cdots \to \Sigma_{1,E} \to \Sigma_{0,E} \to \Omega_{\mathrm{Dr},E} \tag{3.16}$$

exhibits $\{\Sigma_{m,E}\}_m$ as a pro-étale $\overline{D}_L^\times$-torsor over $\Omega_{\mathrm{Dr},E}$.

Lemma 3.20. *There exists an affinoid open subspace $U \subset \Sigma_{m,E}$ such that Assumption 3.10 is satisfied for the $\overline{G}$-action on $\Sigma_{m,E}$. The same statement holds for the $\overline{G}$-action on $\Omega_{\mathrm{Dr},E}$.*

Proof. For $\Omega_{\mathrm{Dr},E}$, this is a direct consequence of Lemma 3.18. For $\Sigma_{m,E}$, the proof is similar to Lemma 3.18, using the building and the discussions in [RZ96, § 3.71], or take the preimage of the affinoids in $\Omega_{\mathrm{Dr},E}$. □

Therefore, Theorem 3.16 applies to $\overline{G}$-equivariant vector bundles on $\Sigma_{m,E}$, e.g. $\mathcal{O}_{\Sigma_{m,E}}$ and the sheaf of differentials $\Omega^1_{\Sigma_{m,E}}$. The following example will be used for our application to the $\mathrm{GL}_2(\mathbb{Q}_p)$-case in § 5.3. Let $\pi_m : \Sigma_{m,E} \to \Omega_{\mathrm{Dr},E}$ be the period maps. Since π_m is $\overline{G}$-equivariant, $\pi_{m,*}\mathcal{O}_{\Sigma_{m,E}}$ is naturally a $\overline{G}$-equivariant sheaf on $\Omega_{\mathrm{Dr},E}$.

Definition 3.21. Let τ be an irreducible smooth representation of $\overline{D}_L^\times$ over E and $\tau^\vee = \mathrm{Hom}_E(\tau, E)$ be the dual representation. We define a $\overline{G}$-equivariant $\mathcal{O}_{\Omega_{\mathrm{Dr},E}}$-module

$$\mathcal{F}_\tau := \mathrm{Hom}_{D_L^\times}(\tau^\vee, \varinjlim_m \pi_{m,*}\mathcal{O}_{\Sigma_{m,E}})$$

which is a vector bundles by Lemma 3.22 below

Lemma 3.22. *The equivariant sheaf $\mathcal{F}_\tau$ in Definition 3.21 is a vector bundle on $\Omega_{\mathrm{Dr},E}$. Moreover, there is an isomorphism $\mathcal{F}_\tau^\vee \simeq \mathcal{F}_{\tau^\vee}$ of equivariant vector bundles.*

Proof. The group $\overline{D}_L^\times$ is compact, hence the action of $\overline{D}_L^\times$ on $\varinjlim_m \pi_{m,*}\mathcal{O}_{\Sigma_{m,E}}$ is semisimple and $\overline{D}_L^\times$ acts on τ through a finite quotient $\overline{D}_L^\times/(1+\varpi_L^k\mathcal{O}_{D_L})$ for some $k \geq 1$. Note that the map π_m is finite étale, since the maps $\breve{\mathcal{M}}_m/\varpi_L^{\mathbb{Z}} \to \Omega_{\mathrm{Dr},\breve{E}}$ is and π_m is its descent (using [Con99, Lem. 1.1.5]). Hence each $\pi_{m,*}\mathcal{O}_{\Sigma_{m,E}}$ is a vector bundle on $\Omega_{\mathrm{Dr},E}$. Then $\mathcal{F}_\tau$ is a direct summand of the $\overline{G}$-equivariant vector bundle $\pi_{k,*}\mathcal{O}_{\Sigma_{k,E}} \simeq (\pi_{m,*}\mathcal{O}_{\Sigma_{m,E}})^{(1+\varpi_L^k\mathcal{O}_{D_L})}, m \geq k$, thus is a vector bundle.

The trace pairing [Sta25, Tag 0BVH] for the finite étale map π_k induces a $(\overline{G}\times\overline{D}_L^\times)$-equivariant perfect pairing. Hence $\pi_{k,*}\mathcal{O}_{\Sigma_{k,E}} \simeq (\pi_{k,*}\mathcal{O}_{\Sigma_{k,E}})^\vee$ equivariantly and $\mathcal{F}_\tau^\vee \simeq \mathcal{F}_{\tau^\vee}$. □

By Lemma 3.22 and Theorem 3.16, we get

$$\mathbb{D}_{\mathrm{BZ}}(H_c^{n-1}(\Omega_{\mathrm{Dr},E}, \mathcal{F}_\tau)) \simeq H_c^{n-1}(\Omega_{\mathrm{Dr},E}, \mathcal{F}_{\tau^\vee} \otimes \Omega^{n-1}_{\Omega_{\mathrm{Dr},E}})[1-n] \tag{3.17}$$

as $\overline{G}$-representations.

Remark 3.23. The $\overline{G}$ or $\mathrm{GL}_n(L)$ representations in (3.17) should play a role in the p-adic Langlands correspondence and the p-adic Jacquet–Langlands correspondence, see § 5 for the example when $n = 2, L = \mathbb{Q}_p$. For application, we need remove the following restrictions on the above discussion.

(1) The Bernstein–Zelevinsky dual in (3.17) is for $\overline{G}$-representations rather than $\mathrm{GL}_n(L)$-representations. This can be resolved as follows. Let $G = \mathrm{GL}_n(L)$. We have isomorphisms $\mathcal{D}^{la}(G,E) \otimes^{L\square}_{E[\varpi_L^{\mathbb{Z}}]} E \simeq \mathcal{D}^{la}(\overline{G}, E)$ and $R\mathrm{Hom}_{E[\varpi_L^{\mathbb{Z}}]}(E, \mathcal{C}_c^{la}(G,E)) \simeq \mathcal{C}_c^{la}(\overline{G}, E)[-1]$. Hence

$$\begin{aligned} R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(V, \mathcal{C}_c^{la}(G,E)) &\simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(\overline{G},E)}(V, R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(\mathcal{D}^{la}(\overline{G},E), \mathcal{C}_c^{la}(G,E))) \\ &\simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(\overline{G},E)}(V, R\underline{\mathrm{Hom}}_{E[\varpi_L^{\mathbb{Z}}]}(E, \mathcal{C}_c^{la}(G,E))) \\ &\simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(\overline{G},E)}(V, \mathcal{C}_c^{la}(\overline{G},E))[-1]. \end{aligned} \tag{3.18}$$

Similar result holds Replacing $\overline{G}$ by $\mathrm{PGL}_n(L)$ using that

$$\begin{aligned} \mathcal{D}^{la}(G,E) \otimes^{L\square}_{\mathcal{D}^{la}(L^\times,E)} E &\simeq \mathcal{D}^{la}(\mathrm{PGL}_n(L), E) \\ R\mathrm{Hom}_{\mathcal{D}^{la}(L^\times,E)}(E, \mathcal{C}_c^{la}(G,E)) &\simeq \mathcal{C}_c^{la}(\mathrm{PGL}_n(L), E)[-1]. \end{aligned}$$

These isomorphisms can be obtained using the resolutions of the trivial representation in [RJRC22, Thm. 5.8].

(2) One may want to consider $D_L^\times$-representations that do not factor through $\overline{D}_L^\times$, so that $\varpi_L^{\mathbb{Z}}$ acts non-trivially on $H_c^{n-1}(\Omega_{\mathrm{Dr},E}, \mathcal{F}_\tau)$. In practice, this can be resolved by twisting a character $G = \mathrm{GL}_n(L) \xrightarrow{\det} L^\times \xrightarrow{\chi} E^\times$ (after enlarging E) and noticing that

$$\mathbb{D}_{\mathrm{BZ}}(V \otimes \chi\circ\det) \simeq \mathbb{D}_{\mathrm{BZ}}(V) \otimes \chi^{-1}\circ\det.$$

One can also consider the duality theory of categories of G-equivariant vector bundles and G-representations with fixed central characters, as in Appendix B.

4. Finiteness for locally analytic representations

Let G be a p-adic Lie group. We define a notion of "finite presentation" for locally analytic representations of G (Definition 4.1) and prove finiteness result for certain Ext groups (Theorem 4.15). Let E be a finite extension of $\mathbb{Q}_p$.

4.1. **Representations of compact presentation.** We define locally analytic representations of compact presentation and show that compactly supported coherent cohomologies of the Drinfeld spaces give such representations. We also show that locally analytic representations of compact presentation are prim (Theorem 4.10).

Definition 4.1. Let G be a p-adic Lie group. Let $V \in \mathrm{Rep}^{la}_{E_\square}(G)$. A *compact presentation* of V consists of a sequence $V^\bullet_{r_k}, k \in \mathbb{N}$ of complexes in $\mathrm{Rep}^{la}_{E_\square}(G)^\heartsuit$ with transition maps $\gamma_k : V^\bullet_{r_k} \to V^\bullet_{r_{k+1}}$ between complexes:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \\
V^\bullet_{r_k}: & \cdots \to \bigoplus_{j=1}^{l_i} \text{c-Ind}_{K_{i,j}}^G W_{r_k,i,j} & \longrightarrow & \bigoplus_{j=1}^{l_{i+1}} \text{c-Ind}_{K_{i+1,j}}^G W_{r_k,i+1,j} & \to \cdots & & \\
{\scriptstyle\gamma_k}\downarrow & & \downarrow{\scriptstyle\gamma_k^i} & & \downarrow{\scriptstyle\gamma_k^{i+1}} & & \\
V^\bullet_{r_{k+1}}: & \cdots \to \bigoplus_{j=1}^{l_i} \text{c-Ind}_{K_{i,j}}^G W_{r_{k+1},i,j} & \to & \bigoplus_{j=1}^{l_{i+1}} \text{c-Ind}_{K_{i+1,j}}^G W_{r_{k+1},i+1,j} & \to \cdots & & \\
 & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & &
\end{array}$$

such that there are quasi-isomorphisms $V \simeq V^\bullet_{r_k}$ for all k satisfying the following properties:

(1) The complexes $V^\bullet_{r_k}$ are uniformly bounded: there exist integers $a \le b$ such that $V^i_{r_k} = 0$ for all $i \notin [a,b], k \in \mathbb{N}$.
(2) For each $i \in [a,b]$, there exists a finite set of compact open subgroups $K_{i,1}, \cdots, K_{i,l_i} \subset G$ such that for each $k \in \mathbb{N}$, there exist locally analytic representations $W_{r_k,i,j}, j = 1, \cdots, l_i$ of $K_{i,j}$ with an isomorphism

$$V^i_{r_k} \simeq \bigoplus_{j=1}^{l_i} \text{c-Ind}_{K_{i,j}}^G W_{r_k,i,j}. \tag{4.1}$$

(3) For all $k \in \mathbb{N}, i \in [a,b], j \in [1,l_i]$, the underlying solid vector spaces $W_{r_k,i,j}$ are Smith spaces.
(4) For all k,i,j, there are *trace-class* morphisms $\gamma_{k,i,j} : W_{r_k,i,j} \to W_{r_{k+1},i,j}$ of $E_\square$-spaces, which are $K_{i,j}$-equivariant and induce $\gamma^i_k : V^i_{r_k} \to V^i_{r_{k+1}}$ via compact induction and (4.1).
(5) the morphisms $\gamma_k = \gamma_k^\bullet$ are quasi-isomorphism for all k.

An object $V \in \mathrm{Rep}^{la}_{E_\square}(G)$ is said to be *of compact presentation* or *compactly presented* if it admits a compact presentation as above.

Remark 4.2. The word "compact" signifies that the Smith spaces W are compact objects in $\mathrm{Mod}(E_\square)$, the maps $\gamma_{r_k,i,j}$ are dual to compact operators of Banach spaces. Moreover, the underlying E-space of a compactly presented representation is similar to a compact type space (being nuclear by Theorem 4.10).

Let L be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_L$ and let $\mathbb{G}$ be a reductive group over L with $G = \mathbb{G}(L)$. We give examples of compactly presented G-representations.

Example 4.3. Every finitely generated smooth E-representation of G is of finite presentation, i.e. is quasi-isomorphic to a bounded complex with terms finite direct sums of representations of the form $\text{c-Ind}_K^G W$ where W is finite-dimensional and $K \subset G$ is open compact. For admissible representations, this follows from the Schneider–Stuhler resolution [SS97, Prop. II.2.1, Thm II.3.1] and for general see [BL23, Cor. 1.4]. Since finite-dimensional spaces are Smith and their identity maps are trace-class, finitely presented representations are of compact presentation.

Example 4.4. The work of Kohlhaase–Schraen [KS12] (slightly extended in [SW25]) shows that locally analytic principal series representations of G are of compact presentation if $\mathbb{G}$ is split. Fix a splitting model of $\mathbb{G}$ with a Borel subgroup and maximal torus $\mathbb{B} \supset \mathbb{T}$ over $\mathcal{O}_L$. Let $T = \mathbb{T}(L), B = \mathbb{B}(L)$. Let $\chi : T \to E^\times$ be a continuous (locally analytic) character and $\mathrm{Ind}_B^G \chi$ a locally analytic principal series. Let $I \subset \mathbb{G}(\mathcal{O}_L)$ be the Iwahori subgroup, the preimage of $\mathbb{B}(\mathcal{O}_L/\varpi_L) \subset \mathbb{G}(\mathcal{O}_L/\varpi_L)$. Fix good analytic open normal subgroups $\cdots \subset I_{k+1} \subset I_k \subset \cdots \subset I$ as in [Eme06, Def. 4.1.3, Prop. 4.1.6] which form a neighbourhood basis of the identity and admit

rigid analytic Iwahori decompositions. For k large enough such that χ is rigid analytic on $T \cap I_k$ [Eme17, § 3.4], consider the Banach I-representation

$$\mathcal{A}_k := (\mathrm{Ind}_B^G \chi(BI))^{I_k\text{-an}} \subset \mathrm{Ind}_B^G \chi$$

of I_k-rigid analytic functions supported on BI. Then [KS12, Thm. 2.5] shows that there is a resolution of the form

$$V_{r_k}^\bullet := \wedge^\bullet E^d \otimes_E \text{c-Ind}_I^G \mathcal{A}_k \xrightarrow{\sim} \mathrm{Ind}_B^G \chi \tag{4.2}$$

where d is the rank of T (see [SW25, Prop. 5.2] for the solid version). For any k, the inclusion $\mathcal{A}_k \to \mathcal{A}_{k+1}$ is E-compact and induces a quasi-isomorphism $V_{r_k}^\bullet \xrightarrow{\sim} V_{r_{k+1}}^\bullet$. Hence the collection $\{V_{r_k}^\bullet\}_k$ satisfies the requirements in Definition 4.1, except for that $\mathcal{A}_k$ are Banach rather than Smith. After changing suitably the choice of $\mathcal{A}_k$, e.g. [SW25, Lem. 2.17], we can take $\mathcal{A}_k$ Smith and the map $\mathcal{A}_k \to \mathcal{A}_{k+1}$ trace-class. Then we get a compact presentation of $\mathrm{Ind}_B^G \chi$.

It is also shown in [SW25] that representations constructed using Orlik–Strauch's functor $\mathcal{F}_B^G$ are of compact presentation.

Example 4.5. Agrawal–Strauch generalized the Schneider–Stuhler complexes [SS97] for locally analytic representations in [AS24]. For an admissible locally analytic representation V of G, they constructed a bounded complex $\mathcal{S}_\bullet(V)$ [AS24, Def. 4.13] whose terms are finte sums of representations of the form $\text{c-Ind}_{P_\sigma^\dagger}^G \underline{\mathrm{Hom}}_{U_\sigma}(\mathcal{D}_r(U_\sigma, E), V)$ [AS24, (4.16)] where $U_\sigma \subset P_\sigma^\dagger$ are open compact (modulo center) subgroups of G indexed by facets of the Bruhat–Tits building and $\mathcal{D}_r(U_\sigma, E), r \in (1/p, 1)$ are the usual Banach distribution algebras of Schneider–Teitelbaum [ST03, p.162]. The E-spaces $\underline{\mathrm{Hom}}_{U_\sigma}(\mathcal{D}_r(U_\sigma, E), V)$ are Smith [AS24, Prop. 3.25, Prop. 3.20]. It's likely that the maps $\underline{\mathrm{Hom}}_{U_\sigma}(\mathcal{D}_r(U_\sigma, E), V) \to \underline{\mathrm{Hom}}_{U_\sigma}(\mathcal{D}_{r'}(U_\sigma, E), V)$ are trace-class. If the natural map $\mathcal{S}_\bullet(V) \to V$ is a quasi-isomorphism, these resolutions varying r would give a compact presentation of V. The last assumption already holds in some situations, see [AS24, § 4.18].

Mikami gave the following example of a compactly presented representation that is not of finite presentation for the simplest non-compact group $\mathbb{Z}$ in [Mik26b, Exam. A.2].

Example 4.6. Let $G = \mathbb{Z}$, then by the Cartier duality we have $\mathrm{Rep}_{\mathbb{Q}_{p,\square}}^{la}(G) \simeq D_\square(\mathbb{Q}_p[T^\pm])$, cf. [RJRC23, Thm. 4.4.4]. Let $W = (\prod_{n\geq 0} \mathbb{Z}_p)[\frac{1}{p}]$ which is Smith and $\text{c-Ind}_{\{0\}}^{\mathbb{Z}} W = \mathbb{Q}_p[T^\pm] \otimes_{\mathbb{Q}_p}^\square W$. Let $\gamma \in \mathrm{End}_{\mathbb{Q}_p}(W)$ be the map $(a_0, a_1, a_2, \cdots) \mapsto (0, a_0, pa_1, \cdots)$ and

$$\alpha : \text{c-Ind}_{\{0\}}^{\mathbb{Z}} W \to \text{c-Ind}_{\{0\}}^{\mathbb{Z}} W, \quad 1 \otimes a \mapsto 1 \otimes a - T^{-1} \otimes \gamma(a)$$

for $a \in W$. Let $V := \mathrm{coker}(\alpha)$. Then $V \simeq \mathrm{colim}(W \xrightarrow{\gamma} W \xrightarrow{\gamma} \cdots)$ where T acts on W by γ. The map γ is trace-class. Hence the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{c-Ind}_{\{0\}}^{\mathbb{Z}} W & \xrightarrow{\alpha} & \text{c-Ind}_{\{0\}}^{\mathbb{Z}} W & \longrightarrow & V & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \text{c-Ind}_{\{0\}}^{\mathbb{Z}} \gamma} & & \downarrow{\scriptstyle \text{c-Ind}_{\{0\}}^{\mathbb{Z}} \gamma} & & \downarrow{\scriptstyle \simeq\, T\times} & & \\ 0 & \longrightarrow & \text{c-Ind}_{\{0\}}^{\mathbb{Z}} W & \xrightarrow{\alpha} & \text{c-Ind}_{\{0\}}^{\mathbb{Z}} W & \longrightarrow & V & \longrightarrow & 0. \end{array}$$

shows that V is of compact presentation. We note that since $\mathbb{Q}_p[T^\pm]$ is nuclear over $\mathbb{Q}_p$, $V \in D_\square(\mathbb{Q}_p[T^\pm])$ is dualizable if and only if it is compact and nuclear if and only if it is compact with its underlying $\mathbb{Q}_p$-vector space nuclear, cf. Theorem 4.10.

As another example, we show that compactly supported coherent cohomologies of the Drinfeld spaces are of compact presentation.

Proposition 4.7. *Let $G = \mathrm{PGL}_n(L)$ and let $\mathcal{F}$ be a G-equivariant vector bundle on the Drinfeld space $\Omega_{\mathrm{Dr},E}$ of dimension $n-1$ as in § 3.4. Then the compactly supported cohomology $H_c^{n-1}(\Omega_{\mathrm{Dr},E}, \mathcal{F})$ is a locally analytic representation of G of compact presentation in the sense of Definition 4.1.*

Proof. Let $U_{r_k} \subset \Omega_{\mathrm{Dr},E}$ be the affinoid open subspaces in Lemma 3.18 and K the open compact subgroup of G that stabilizes all U_{r_k}. We write $K_\alpha = g_0 K g_0^{-1} \cap \cdots \cap g_n K g_n^{-1}$ as in §3.3. Take $\alpha \in G\backslash(G/K)^{n+1}$. By Proposition 2.28, $R\Gamma_c(U_{\alpha,r_k}, \mathcal{F}) \simeq H_c^{n-1}(U_{\alpha,r_k}, \mathcal{F})[1-n]$ and $H_c^{n-1}(U_{\alpha,r_k}, \mathcal{F})$

is a Smith space representation of E. The resolution (3.12) shows that $R\Gamma_c(\Omega_{\mathrm{Dr},E},\mathcal{F})$ can be computed using the covering $\{g.U_{r_k}\}_{g\in G/K}$ and is quasi-isomorphic to the following (Čech) complex

$$\begin{aligned}(4.3)\qquad &\cdots\to\oplus_{\alpha=(1,g_1,\cdots,g_i)\in G\backslash(G/K)^{i+1}}\text{c-Ind}_{K_\alpha}^G H_c^{n-1}(U_{r_k,\alpha},\mathcal{F})\to\cdots\\ &\to\oplus_{(1,g)\in G\backslash(G/K)^2}\text{c-Ind}_{K\cap gKg^{-1}}^G H_c^{n-1}(g.U_{r_k}\cap U_{r_k},\mathcal{F})\to\text{c-Ind}_K^G H_c^{n-1}(U_{r_k},\mathcal{F}).\end{aligned}$$

For each $i\geq 0$, define a set

$$\mathcal{T}_i:=\{S\subset G/K\mid |S|=i+1\}.$$

Formally, by the proof of [Ser55, §3, Prop. 2], the complex (4.3) is quasi-isomorphic to the following alternating Čech complex (a direct summand of (4.3))

$$\begin{aligned}(4.4)\qquad V_{r_k}^\bullet:\cdots&\to\oplus_{\alpha\in G\backslash\mathcal{T}_i}\text{c-Ind}_{K_\alpha^{\mathrm{alt}}}^G\left(H_c^{n-1}(U_{r_k,\alpha},\mathcal{F})\otimes_E\mathrm{sgn}_\alpha\right)\to\cdots\\ &\to\oplus_{\alpha\in G\backslash\mathcal{T}_1}\text{c-Ind}_{K_\alpha^{\mathrm{alt}}}^G\left(H_c^{n-1}(U_{r_k,\alpha},\mathcal{F})\otimes_E\mathrm{sgn}_\alpha\right)\to\text{c-Ind}_K^G H_c^{n-1}(U_{r_k},\mathcal{F}).\end{aligned}$$

Here in (4.4), for $\alpha=\{g_0,\cdots,g_i\}\in\mathcal{T}_i$, a subset of G/K of size $i+1$, we let

$$K_\alpha^{\mathrm{alt}}:=\{g\in G\mid g.\alpha=\alpha\subset G/K\},$$

which is an open compact subgroup of G containing $K_\alpha=\cap_{a=0}^i g_aKg_a^{-1}$ and stabilizing $\cap_{a=0}^i g_a.U_{r_k}$, and $\mathrm{sgn}_\alpha(g)=\mathrm{sgn}(\sigma)$ if $g.(g_0,\cdots,g_i)=(g_{\sigma(0)},\cdots,g_{\sigma(i)})$ for $g\in K_\alpha^{\mathrm{alt}}$ where $\mathrm{sgn}(\sigma)$ is the sign of the permutation.

There exists a finite subset $\mathcal{S}\subset G/K$ such that the set $g.U_{r_k}\cap U_{r_k}=\emptyset$ if $g\notin\mathcal{S}$ for all k. For each i, every element in $G\setminus\{\alpha\subset G/K\mid U_{r_k,\alpha}\neq\emptyset,\alpha\in\mathcal{T}_i\}$ has at least one representative in

$$\mathcal{T}_i':=\{S\subset\mathcal{S}\mid |S|=i+1\}.$$

Choose representatives $\alpha_{i,1},\cdots,\alpha_{i,l_i}\in\mathcal{T}_i'$ for elements in $G\backslash\mathcal{T}_i$ if such representatives exist. For $i\in[-|\mathcal{S}|-1,0], j\in[1,l_i]$, set $K_{i,j}=K_{\alpha_{-i,j}}^{\mathrm{alt}}$, and $W_{r_k,i,j}:=H_c^{n-1}(U_{r_k,\alpha_{-i,j}},\mathcal{F})\otimes\mathrm{sgn}_{\alpha_{-i,j}}$. Let $\gamma_{k,i,j}:W_{r_k,i,j}\to W_{r_{k+1},i,j}$ be the natural map induced by the inclusion $U_{r_k,\alpha_{i,j}}\hookrightarrow U_{r_{k+1},\alpha_{i,j}}$. Since $\Omega_{\mathrm{Dr},E}$ is separated, $U_{r_k,\alpha}$ are affinoid [BGR84, § 9.6.1, Prop. 6]. Since U_{r_k} is relatively compact in $U_{r_{k+1}}$, it follows easily from the definition in [BGR84, § 9.6.2] that $U_{r_k,\alpha_{i,j}}$ is relatively compact in $U_{r_{k+1},\alpha_{i,j}}$. Hence $\gamma_{k,i,j}$ is trace-class by Corollary 2.29. We conclude that the resolutions $V_{r_k}^\bullet$ give a compact presentation of $H_c^{n-1}(\Omega_{\mathrm{Dr},E},\mathcal{F})$. □

Lemma 4.8. *Let K be a compact p-adic Lie group and let W be a soid locally analytic representation whose underlying solid E-vector space is Smith. Then W is (h^+)-analytic for some $h>0$ (see the proof for the meaning of (h^+)-analytic). The associated Banach space $W^B\to W$ (see (2.9)) carries a unique locally analytic K-action for which $W^B\to W$ is $\mathcal{D}^{la}(K,E)$-linear.*

Proof. We fix certain notation which will also be used later. We fix a uniform pro-p subgroup of K so that we have the distribution algebras $\mathcal{D}^{(h)}(K,E),\mathcal{D}^{(h^+)}(K,E)$ and $\mathcal{D}_{(h)}(K,E)$ in [RJRC22, § 4.1.4] for $h\in\mathbb{Q}_{>0}$. Here $\mathcal{D}_{(h)}(K,E)$ is Banach, $\mathcal{D}^{(h)}(K,E)$ is Smith and $\mathcal{D}^{(h^+)}(K,E)=\varinjlim_{h'>h}\mathcal{D}^{(h)}(K,E)=\varinjlim_{h'>h}\mathcal{D}_{(h)}(K,E)$ is LS. For $h'>h$, the map $\mathcal{D}^{(h')}(K,E)\to\mathcal{D}^{(h)}(K,E)$ is trace-class and there are natural maps $\mathcal{D}_{(h')}(K,E)\to\mathcal{D}^{(h)}(K,E)\to\mathcal{D}_{(h)}(K,E)$ ([RJRC22, Cor. 4.18]). A locally analytic representation W of K is (h^+)-analytic if the action of $\mathcal{D}^{la}(K,E)$ on W extends to $\mathcal{D}^{(h^+)}(K,E)$ and we write $W^{(h^+)-\mathrm{an}}$ for the space of (h^+)-analytic vectors of W, see [RJRC22, Def. 4.29, Thm. 4.36].

By [RJRC22, Prop. 4.41], if W is Smith, then W is (h^+)-analytic for some h, hence is a $\mathcal{D}_{(h)}(K,E)$-module for some h. Applying the trace-class functor (2.8) to the action map

$$\mathcal{D}_{(h)}(K,E)\otimes_E^\square W\to W$$

we obtain a map

$$\mathcal{D}_{(h)}(K,E)^{\mathrm{tr}}\otimes_E^\square W^{\mathrm{tr}}\to(\mathcal{D}(K^{(h)},E)\otimes_E^\square W)^{\mathrm{tr}}\to W^{\mathrm{tr}}$$

Since $\mathcal{D}_{(h)}(K,E)$ is Banach, $\mathcal{D}_{(h)}(K,E)=\mathcal{D}_{(h)}(K,E)^{\mathrm{tr}}$ and we obtain a map $\mathcal{D}_{(h)}(K,E)\otimes_E^\square W^B\to W^B$. It is formal to check that this makes W^B a $\mathcal{D}_{(h)}(K,E)$-module, hence it is a locally analytic K-representation by [RJRC22, Thm. 4.36]. □

Let G be a p-adic Lie group. Let $\mathrm{Rep}_{E_\square}^{la}(G)^\omega\subset\mathrm{Rep}_{E_\square}^{la}(G)$ be the full subcategory consisting of compact objects [Sta25, Tag 07LS] (see also [Lur17, Prop. 1.4.4.1(3)]).

Lemma 4.9. *Let $K\subset G$ be a compact open subgroup.*

(1) *Let* $W \in \mathrm{Rep}^{la}_{E_\square}(K)^\heartsuit$ *with underlying solid* E*-vector space Smith. Then* $\text{c-Ind}_K^G W$ *is a compact object of* $\mathrm{Rep}^{la}_{E_\square}(G)$.

(2) *There exists a set of compact generators of* $\mathrm{Rep}^{la}_{E_\square}(G)$ *consisting of representations of the form* $\text{c-Ind}_K^G W$ *where* W *is Smith. Any object in* $\mathrm{Rep}^{la}_{E_\square}(G)^\omega$ *is a retract of a finite complex of finite sums of these compact generators.*

(3) *If* $V \in \mathrm{Rep}^{la}_{E_\square}(G)^\omega$*, then* $\mathbb{D}_{\mathrm{BZ}}(V)$ *is locally analytic and its underlying solid* E*-vector space is nuclear.*

Proof. (1) By [Mik26a, Prop. 2.37(2)], the statement holds if $G = K$. The general case follows from the Frobenius reciprocity.

(2) The Grothendieck abelian category $\mathrm{Rep}^{la}(G)^\heartsuit$ has a set of compact generators $\text{c-Ind}_K^G W$ where W are Smith spaces of the form $\mathcal{D}^{(h)}(K,E) \otimes_{E_\square} E_\square[S], S \in \mathrm{ExtDisc}$, cf. the proof of [RJRC23, Prop. 3.3.6] where it is shown that $\mathrm{Rep}^{la}(G)^\heartsuit$ is generated by $\text{c-Ind}_K^G W^+$ for W^+ of the form

$$\mathcal{D}^{(h^+)}(K,E) \otimes_{E_\square} E_\square[S] = \mathrm{colim}_{h'>h} \mathcal{D}^{(h')}(K,E) \otimes_{E_\square} E_\square[S], \quad S \in \mathrm{ExtDisc}.$$

Indeed, by [RJRC22, Thm. 4.36(1)], one has

$$R\mathrm{Hom}_{\mathcal{D}^{la}(K,E)}(\mathcal{D}^{(h)}(K,E) \otimes_{E_\square} E_\square[S], C) \simeq C^{R(h)-\mathrm{an}}(S)$$

for any $C \in \mathrm{Rep}^{la}_{E_\square}(G)$ and the functor $(-)^{R(h)-\mathrm{an}}$ preserves colimits. The last assertion follows from [Sta25, Tag 0948].

(3) This follows from (2) and Proposition 3.2: retracts, extensions and colimits of nuclear spaces are nuclear [And21, Prop. 5.33, Lem. 5.40] and $\mathbb{D}_{\mathrm{BZ}}(\text{c-Ind}_K^G W) = \text{c-Ind}_K^G W^\vee$ is nuclear if W is Smith. □

Theorem 4.10. *Let* G *be a* p*-adic Lie group. Let* $V \in \mathrm{Rep}^{la}_{E_\square}(G)$*. We view* V *as an object in* $D([*/G^{la}])$ *via* (3.5)*. Let* $f : [*/G^{la}] \to *$ *be the natural map.*

(1) *The representation* V *is* f*-prim* [HM24, Def. 4.4.1] *if and only if* V *is in* $\mathrm{Rep}^{la}_{E_\square}(G)^\omega$ *and the underlying solid* E*-space of* V *is nuclear. Moreover, if* V *is* f*-prim, the prim dual* $\mathrm{PD}_f(V)$ *corresponds to the Bernstein–Zelevinsky dual* $\mathbb{D}_{\mathrm{BZ}}(V)$ *and the natural morphism* $V \to \mathbb{D}_{\mathrm{BZ}}(\mathbb{D}_{\mathrm{BZ}}(V))$ *is an isomorphism.*

(2) *If* V *is of compact presentation, then* V *is* f*-prim.*

Proof. (1) We will use the criterion of prim objects in [HM24, Lem. 4.4.6]. Let $\pi_i, i = 1,2$ be the projection $[*/(G\times G)^{la}] \simeq [*/G^{la}] \times [*/G^{la}] \to [*/G^{la}]$ to the i-th factor and let $\Delta : [*/G^{la}] \to [*/(G\times G)^{la}]$ be the diagonal map. We first show that for any $\Pi \in \mathrm{Rep}^{la}_{E_\square}(G)$ with underlying solid E-space nuclear and $P \in \mathrm{Rep}^{la}_{E_\square}(G)^\omega$ is compact, the natural morphism

$$f_!(\pi_{2*}\underline{\mathrm{Hom}}(\pi_1^* P, \Delta_! \mathbf{1}) \otimes \Pi) \to f_* \underline{\mathrm{Hom}}(P, \Pi) \tag{4.5}$$

in *loc. cit.* (see also [Sch25, Prop. 6.9]) is an isomorphism.

Let $K \subset G$ be an open compact subgroup. By (2) of Lemma 4.9, it suffices to prove that (4.5) is an isomorphism for P of the form $\text{c-Ind}_K^G W$ with W Smith. Write $q : [*/K^{la}] \to [*/G^{la}]$ so that $\text{c-Ind}_K^G W$ corresponds to $q_! W \in D([*/G^{la}])$ and q^* corresponds to the restriction of G-representations to K. Since $\Delta_! : D([*/G^{la}]) \to D([*/(G\times G)^{la}])$ is the compact induction functor as in [HM24, § 5.4], $\Delta_!\mathbf{1}$ corresponds to the representation $\mathcal{C}_c^{la}(G,E)$ and the object $\pi_{2*}\underline{\mathrm{Hom}}(\pi_1^* P, \Delta_!\mathbf{1})$ corresponds to the G-representation $\mathbb{D}_{\mathrm{BZ}}(P)^{Rla}$, which is canonically isomorphic to $q_! W^\vee$ by Proposition 3.2. By the projection formula, we have $q_! W^\vee \otimes \Pi \simeq q_!(W^\vee \otimes q^*\Pi)$. By [Mik26a, Prop. 2.39], we have $(f\circ q)_! = (f\circ q)_*$. Hence the lefthand side of (4.5) for $P = q_! W$ is given by $(f\circ q)_*(W^\vee \otimes \Pi|_K)$ calculated by $R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(K,E)}(E, W^\vee \otimes^L_{E_\square} \Pi|_K)$. On the other hand, the righthand side of (4.5) is given by $R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(E, R\underline{\mathrm{Hom}}_E(P,\Pi)^{Rla}) = R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(P,\Pi) \simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(K,E)}(W, \Pi|_K)$. Then (4.5) becomes the natural isomorphism

$$R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(K,E)}(E, W^\vee \otimes^L_{E_\square} \Pi|_K) \simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(K,E)}(E, R\underline{\mathrm{Hom}}_E(W, \Pi|_K))$$

which follows from that $\Pi|_K$ is nuclear and W is Smith. Hence (4.5) is an isomorphism when P is compact and Π is nuclear.

Now [HM24, Lem. 4.4.6] says that an object $P \in D([*/G^{la}])$ is f-prim if and only if the morphism (4.5) is an isomorphism after applying $\mathrm{Hom}(\mathbf{1}, -)$. Hence P is f-prim if P is compact and nuclear by taking $\Pi = P$. The same lemma implies that $\mathrm{PD}_f(P) = \mathbb{D}_{\mathrm{BZ}}(P)^{Rla} = \mathbb{D}_{\mathrm{BZ}}(P)$ in this case.

Conversely, suppose that V is f-prim. Then *loc. cit.* tells that $\mathrm{PD}_f(V) = \mathbb{D}_{\mathrm{BZ}}(V)^{Rla}$ and [HM24, Lem. 4.4.18] implies that V is compact. Lemma 4.9(3) then implies that $\mathrm{PD}_f(V)$ is nuclear. By [HM24, Lem. 4.4.4], $V \simeq \mathrm{PD}_f(\mathrm{PD}_f(V))$ and $\mathrm{PD}_f(V)$ is f-prim. Hence V is nuclear.

(2) Suppose that V admits a compact presentation $\{V_{r_k}^\bullet\}$. Then V is compact by Lemma 4.9. Consider the trace-class functor $(-)^{\mathrm{tr}}$ in [Bos23b, Def. A.13]. The quasi-isomorphism γ_k factors through a morphism $V_{r_k}^\bullet \to V_{r_{k+1}}^{\bullet,\mathrm{tr}} \to V_{r_{k+1}}^\bullet$. Hence V is a retract of V^{tr}, so V is nuclear and $V \simeq V^{\mathrm{tr}}$. We conclude that V is f-prim by (1). □

Corollary 4.11. *In the situation of Theorem 4.10, if G is compact, then $V \in \mathrm{Rep}_{E_\square}^{la}(G)$ is f-prim if and only if its underlying solid E-vector space is perfect.*

Proof. By [Mik26a, Prop. 2.37], V is compact if and only if the underlying E-space is compact. The statement then follows from that $V \in D(E_\square)$ is nuclear and compact if and only if V is dualizable if and only if V is perfect [And21, Prop. 5.37, Cor. 5.51.1]. See also [RC26b, Cor. 6.2.52]. □

Remark 4.12. By Lemma 4.8, if V admits a compact presentation, then $V \simeq V_{r_k}^{\bullet,\mathrm{tr}}$ is a bounded complex of finite sums of representations of the form $\text{c-Ind}_K^G W$ where W is Banach. However, we do not know whether $\mathbb{D}_{\mathrm{BZ}}(V)$ is of compact presentation, even though the natural map $V \to \mathbb{D}_{\mathrm{BZ}}(\mathbb{D}_{\mathrm{BZ}}(V))$ is an equivalence by the theorem above.

4.2. **Finiteness of** Ext **groups.** We prove the main finiteness result for locally analytic representations of compact presentation (Theorem 4.15).

Lemma 4.13. *Let K be an open compact subgroup of G and let $f : W_{r_0} \to W_{r_1}$ be a K-equivariant morphism of locally analytic representations whose underlying solid E-vector spaces are Smith. We assume that f is trace-class. If $\Pi \in \mathrm{Rep}_{E_\square}^{la}(G)^\heartsuit$ is an admissible locally analytic representation of K, then the induced morphism*

$$R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(K,E)}(W_{r_1}, \Pi) \longrightarrow R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(K,E)}(W_{r_0}, \Pi).$$

in $D(E_\square)$ is represented by a morphism of bounded complexes of Banach spaces which is degreewise compact.

Proof. We follow the notation in the proof of Lemma 4.8. Let

$$\Pi^{R(h^+)-\mathrm{an}} = R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(K,E)}(\mathcal{D}^{(h^+)}(K,E), \Pi)$$

be the derived h^+-analytic vectors of Π ([RJRC22, Def. 4.29, Thm. 4.36]). We claim that $\Pi^{R(h^+)-\mathrm{an}} \simeq \Pi^{(h^+)-\mathrm{an}}$ (cf. [AS24, Prop. 3.27]). Since Π is admissible, $\Pi = R\underline{\mathrm{Hom}}_E(\Pi^\vee, E)^{Rla}$ by [RJRC23, Lem. 4.2.6], and as in the proof of *loc. cit.* (and also in [RJRC22, Prop. 4.48]),

$$\begin{aligned}
\Pi^{R(h^+)-\mathrm{an}} &\simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(K,E)}(\mathcal{D}^{(h^+)}(K,E), R\underline{\mathrm{Hom}}_E(\Pi^\vee, E)) \\
&\simeq R\underline{\mathrm{Hom}}_E(\mathcal{D}^{(h^+)}(K,E) \otimes^{L\square}_{\mathcal{D}^{la}(K,E)} \Pi^\vee, E) \\
&\simeq \underline{\mathrm{Hom}}_E(\mathcal{D}^{(h^+)}(K,E) \otimes^{\square}_{\mathcal{D}^{la}(K,E)} \Pi^\vee, E) \\
&\simeq \Pi^{(h^+)-\mathrm{an}}
\end{aligned}$$

where for the third equality, we used [RJRC23, Cor. 4.2.4] and that $\mathcal{D}^{(h^+)}(K,E) \otimes^{\square}_{\mathcal{D}^{la}(K,E)} \Pi^\vee$ is isomorphic to a finite complex of LS spaces of compact type (see discussions below).

Hence

$$R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(K,E)}(W_{r_k}, \Pi) = R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(K,E)}(W_{r_k}, \Pi^{R(h^+)-\mathrm{an}}) = R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(K,E)}(W_{r_k}, \Pi^{(h^+)-\mathrm{an}}).$$

Choose $h' > h$, then $\underline{\mathrm{Hom}}_E(\mathcal{D}_{(h')}(K,E) \otimes^{\square}_{\mathcal{D}(K,E)} \Pi^\vee, E)$ is h^+-analytic. Since $\Pi^\vee$ is coadmissible over $\mathcal{D}^{la}(K,E)$, $\mathcal{D}_{(h')}(K,E) \otimes^{\square}_{\mathcal{D}(K,E)} \Pi^\vee$ is finitely generated over $\mathcal{D}_{(h')}(K,E)$. By [ST03, Thm. 8.9], the distribution algebra $\mathcal{D}_{(h')}(K,E)(*)$ is Auslander-regular with finite global projective dimension. Thus we may choose a resolution of $\mathcal{D}_{(h')}(K,E) \otimes^{\square}_{\mathcal{D}(K,E)} \Pi^\vee$ by finite projective $\mathcal{D}_{(h')}(K,E)$-modules $M_{(h'),d} \to \cdots \to M_{(h'),1} \to M_{(h'),0}$. Tensoring with the LS algebras $\mathcal{D}^{(h^+)}(K,E)$ [RJRC22, Thm. 4.45], we obtain a resolution of $\Pi^{(h^+)-\mathrm{an},\vee}$ by $M_\bullet^{(h^+)} = \mathcal{D}^{(h^+)}(K,E) \otimes_{\mathcal{D}_{(h')}(K,E)} M_{(h'),\bullet}$.

Taking the E-dual which is exact for these LS spaces [RJRC22, Thm. 3.40], we see $\Pi^{(h^+)-\mathrm{an}}$ is isomorphic to a complex $M_\bullet^{(h^+),\vee}$ where each $M_i^{(h^+),\vee}$ is a direct summand of

$$\mathcal{D}^{(h^+)}(K,E)^{\vee,\oplus n_i} \simeq \mathcal{C}^{(h^+)-\mathrm{an}}(K,E)^{\oplus n_i}.$$

Let h be large enough such that $W_{r_k}, k=0,1$ are (h^+)-analytic. As in the proof of Proposition 3.2, $R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(K,E)}(W_{r_k}, M_i^{(h^+),\vee})$ is a direct summand of

$$R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(K,E)}(W_{r_k}, \mathcal{C}^{(h^+)-\mathrm{an}}(K,E)^{\oplus n_i}) \simeq W_{r_k}^{\vee,\oplus n_i}.$$

Hence

$$R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(K,E)}(W_{r_k}, M_i^{(h^+),\vee}) = \underline{\mathrm{Hom}}_{\mathcal{D}^{la}(K,E)}(W_{r_k}, M_i^{(h^+),\vee})$$

and the induced map $\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(K,E)}(W_{r_1}, M_i^{(h^+),\vee}) \to \underline{\mathrm{Hom}}_{\mathcal{D}^{la}(K,E)}(W_{r_0}, M_i^{(h^+),\vee})$ is a compact morphism between Banach spaces (as a direct summand of the compact morphism $W_{r_1}^{\vee,\oplus n_i} \to W_{r_0}^{\vee,\oplus n_i}$). The result follows. □

The following lemma is a derived variant of Proposition 2.33.

Lemma 4.14. *Let $C_{k,i}, k, i = 0, 1, \cdots, N$ be objects in $D(E_\square)$ with morphisms $g_{k,i} : C_{k,i} \to C_{k,i+1}$ and $\gamma_{k,i} : C_{k,i} \to C_{k+1,i}$. Assume that $C_{k,0} = 0$ and write $C_k := C_{k,N}$ for all k. Let $\mathrm{gr}_{k,i} := \mathrm{Cone}(g_{k,i}) \in D(E_\square)$ for $i \leq N-1$. We assume that for each k, i, there is a homotopy $g_{k+1,i} \circ \gamma_{k,i} \simeq \gamma_{k,i+1} \circ g_{k,i}$ such that the induced map $\gamma_{k,i} : \mathrm{gr}_{k,i} \to \mathrm{gr}_{k+1,i}$ in $D(E_\square)$ can be represented by a degreewise compact morphism between bounded complexes of Banach E-spaces.*

If the composition $\gamma = \gamma_{N-1,N} \circ \cdots \circ \gamma_{1,N} \circ \gamma_{0,N} : C_0 \to C_N$ is a quasi-isomorphism, then C_0 is a perfect object in $D(E_\square)$.

Proof. An object in $D(E_\square)$ is perfect if and only if C is dualizable if and only if C is nuclear and compact by [CS22, Prop. 9.3] and [And21, Lem. 5.46, Cor. 5.51.1]. Since Banach spaces are nuclear [RJRC22, Prop. 3.29] and by our assumption $\mathrm{gr}_{k,i}$ is isomorphic to a bounded complex of Banach spaces, it is nuclear. It follows from [And21, Lem. 5.40] and an induction on i using the triangles $C_{k,i} \to C_{k,i+1} \to \mathrm{gr}_{k,i} \to$ that $C_{k,i}$ are nuclear for all k, i. We remain to show that C_0 is compact.

For the rest of the proof, we write $D(E_\square)$ for its homotopy category which is a triangulated category. Write $D(E_\square)^\omega$ for the full subcategory of compact objects in $D(E_\square)$ consisting of objects that are isomorphic to a retract of a bounded complex of Smith spaces [And21, Lem. 5.36]. Then $D(E_\square)^\omega$ is a triangulated subcategory of $D(E_\square)$ [And21, Lem. 5.40] and is thick, thus we can consider the triangulated quotient category $Q(E_\square) := D(E_\square)/D(E_\square)^\omega$ [Sta25, Tag 05RA]. Let

$$P : D(E_\square) \to Q(E_\square)$$

be the quotient functor so that $P(C) \simeq 0$ if $C \in D(E_\square)^\omega$ [Sta25, Tag 05RK].

If $f : G_1 \to G_2$ is a degreewise compact morphism between bounded complexes of Banach spaces, then $f^\vee : G_2^\vee \to G_1^\vee$ is a degreewise trace-class morphism between bounded complexes of Smith spaces (applying the exact dual in [RJRC22, Thm. 3.40(1)]) which factors through $G_1^{\vee,\mathrm{tr}}$ [Bos23b, Lem. A.14]. Here $G_1^{\vee,\mathrm{tr}}$ is obtained by applying the trace-class functor $(-)^{\mathrm{tr}}$ [Bos23b, Def. A.13] to $G_1^\vee$ and is a complex of Banach spaces, see the proof of [Bos23a, Lem. A.58]. Hence the map f factors through $G_1^{\vee,\mathrm{tr},\vee} \in D(E_\square)^\omega$ a bounded complex of Smith spaces. Therefore, $P(f) : P(G_1) \to P(G_2)$ is a zero morphism.

Thus, by our assumption, the maps $P(\gamma_{k,i}) : P(\mathrm{gr}_{k,i}) \to P(\mathrm{gr}_{k+1,i})$ are zero for all k, i. Since P is exact, the triangle $P(C_{k+1,i}) \to P(C_{k+1,i+1}) \to P(\mathrm{gr}_{k+1,i}) \to$ is exact. Hence the map $P(\gamma_{k,i+1}) : P(C_{k,i+1}) \to P(C_{k+1,i+1})$ factors through a map $P(C_{k,i+1}) \to P(C_{k+1,i})$ (see the proof of [Sta25, Tag 0149]). Consequently, the composition $P(\gamma) = P(\gamma_{N-1,N}) \circ \cdots \circ P(\gamma_{1,N}) \circ P(\gamma_{0,N}) : P(C_0) \to P(C_N)$ factors through a map $P(C_0) = P(C_{0,N}) \to P(C_{1,N-1}) \to \cdots \to P(C_{N-1,1}) \to P(C_{N,0}) = 0$. In particular, $P(\gamma)$ is a zero morphism. Since γ is an isomorphism by our assumption, so is $P(\gamma)$. This forces $P(C_0) = 0$. Hence $C_0 \in D(E_\square)^\omega$ by [Sta25, Tag 05RK], i.e., C_0 is compact. The result follows. □

Theorem 4.15. *Let V be a locally analytic representation of G of compact presentation in the sense of Definition 4.1, and let $\Pi \in \mathrm{Rep}^{la}(G)^\heartsuit$ be an admissible locally analytic representation of G. Then $R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(V, \Pi)$ is a perfect object in $D(E_\square)$. In particular,*

$$\dim_E \mathrm{Ext}^i_{\mathcal{D}^{la}(G,E)}(V, \Pi) < \infty$$

for every $i \geq 0$.

Proof. Take a compact presentation $V^\bullet_{r_k}, k \in \mathbb{N}$ of V as in Definition 4.1 and write $\sigma_{\leq i}V^\bullet_{r_k}$ for the stupid truncation of $V^\bullet_{r_k}$ so that there are exact sequences $0 \to V^i_{r_k}[-i] \to \sigma_{\leq i}V^\bullet_{r_k} \to \sigma_{\leq i-1}V^\bullet_{r_k} \to 0$ of complexes [Sta25, Tag 0118]. Set

$$\begin{aligned} C_{k,i} &:= R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(\sigma_{\leq i}V^\bullet_{r_k}, \Pi), \\ \mathrm{gr}_{k,i} &:= R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(V^{i+1}_{r_k}[-i-1], \Pi) \end{aligned} \tag{4.6}$$

so that $\mathrm{gr}_{k,i} \simeq \mathrm{Cone}(C_{k,i} \to C_{k,i+1})$. Denote again $\gamma_k : C_{k+1,i} \to C_{k,i}$ for the natural map induced by the truncation of the map $\gamma_k : V_{r_k} \to V_{r_{k+1}}$ and the functor $R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(-, \Pi)$. Then γ_k induces a map $\mathrm{gr}_{k+1,i} \to \mathrm{gr}_{k,i}$.

By the assumptions in Definition 4.1, $V^i_k \simeq \oplus_{j=1}^{n_i} \text{c-Ind}_{K_{i,j}}^G W_{r_k,i,j}$ for some Smith space representations of open compact subgroups $K_{i,j}$ of G. By Frobenius reciprocity, we have

$$R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(V^i_k, \Pi) \simeq \oplus_{j=1}^{n_i} R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(K_{i,j},E)}(W_{r_k,i,j}, \Pi).$$

The map $\gamma^i_k : V^i_k \to V^i_{k+1}$ is induced by maps $\gamma_{k,i,j} : W_{r_k,i,j} \to W_{r_{k+1},i,j}$. Thus the map $\mathrm{gr}_{k+1,i} \to \mathrm{gr}_{k,i}$ is the direct sum of maps

$$R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(K_{i,j},E)}(W_{r_{k+1},i,j}, \Pi) \xrightarrow{R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(K_{i,j},E)}(\gamma_{k,i,j},\Pi)} R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(K_{i,j},E)}(W_{r_k,i,j}, \Pi).$$

By Lemma 4.13, the above maps can be represented by degreewise compact morphisms between bounded complexes of Banach spaces. So the same holds for $\mathrm{gr}_{k+1,i} \to \mathrm{gr}_{k,i}$. Since there exists only finitely many i such that $V^i_{r_k} \neq 0$, $C_{k,i} \simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(V, \Pi)$ and $\gamma_k : C_{k+1,i} \to C_{k,i}$ are quasi-isomorphisms for i large enough and for all k, and $C_{k,i} = 0$ for i small enough. After picking finitely many indices k, i and renumbering them suitably, the assumptions in Lemma 4.14 are satisfied for the collection $\{C_{k,i}\}_{k,i}$. The result follows. □

Remark 4.16. We explain the homological algebra issue underlying the proof of Theorem 4.15 and Lemma 4.14 related to Definition 4.1. Let $V \in \mathrm{Rep}^{la}_{E_\square}(G)$ with a compact presentation $V^\bullet_{r_k}$. We take as example the case where $V^i_{r_k} = 0$ if $i \neq 0, -1$. Applying $R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(-, \Pi)$ to $V^\bullet_{r_k}$ produces a commutative diagram in $D(E_\square)$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{gr}_{1,0} & \xrightarrow{f_{1,0}} & \mathrm{gr}_{1,-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \gamma_{0,0}} & & \downarrow{\scriptstyle \gamma_{0,-1}} & & \\ 0 & \longrightarrow & \mathrm{gr}_{0,0} & \xrightarrow{f_{0,0}} & \mathrm{gr}_{0,-1} & \longrightarrow & 0 \end{array} \tag{4.7}$$

where $\mathrm{gr}_{k,i} = R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(V^i_{r_k}, \Pi)$. Moreover, $R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(V, \Pi) \simeq \mathrm{Cone}(f_{i,0})[-1]$. By Lemma 4.13 the map $\gamma_{0,i} : \mathrm{gr}_{1,i} \to \mathrm{gr}_{0,i}$ can be represented by a degreewise compact morphism between bounded complexes of Banach spaces. If the whole diagram (4.7) could be represented by a commutative diagram of bounded chain complexes of Banach spaces with vertical maps degreewise compact, then the quasi-isomorphism $\gamma_0 : \mathrm{Cone}(f_{1,0}) \to \mathrm{Cone}(f_{0,0})$ would satisfy the assumption in Proposition 2.33 and we may conclude that $R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(V, \Pi)$ is perfect.

However, a priori, we may only assume that (4.7) is represented by a diagram of bounded complexes of Banach spaces which is commutative up to homotopy. Suppose $\mathrm{gr}_{k,i} \simeq \mathrm{gr}^\bullet_{k,i}$ are bounded complexes of Banach spaces and the arrows in (4.7) are chain maps with vertical maps degreewise compact. Then there exists a homotopy $h : \mathrm{gr}^\bullet_{1,0} \to \mathrm{gr}^{\bullet-1}_{0,-1}$ such that $\gamma_{0,-1} \circ f_{1,0} - f_{0,0} \circ \gamma_{0,0} = d \circ h + h \circ d$ so that the quasi-isomorphism $\mathrm{Cone}(f_{1,0}) \xrightarrow{\sim} \mathrm{Cone}(f_{0,0})$ is represented by a chain map $\gamma^\bullet_0$ with $\gamma^i_0 = \begin{pmatrix} \gamma^i_{0,-1} & h^{i+1} \\ & \gamma^{i+1}_{0,0} \end{pmatrix} : \mathrm{gr}^i_{1,-1} \oplus \mathrm{gr}^{i+1}_{1,0} \to \mathrm{gr}^i_{0,-1} \oplus \mathrm{gr}^{i+1}_{0,0}$ in degree i [Sta25, Tag 014F]. Since h^i may not be compact, we cannot apply Proposition 2.33 directly for the chain map $\gamma^\bullet_0$. The homotopical issue can be resolved using the third resolution $V^\bullet_{r_2}$. Then one obtains similarly $f_{2,0} : \mathrm{gr}_{2,0} \to \mathrm{gr}_{2,-1}$, a homotopy $h' : \mathrm{gr}^\bullet_{2,0} \to \mathrm{gr}^{\bullet-1}_{1,-1}$ and a chain map $\gamma^\bullet_1 : \mathrm{Cone}(f_{2,0}) \to \mathrm{Cone}(f_{1,0})$ using h'. We observe that the composition $\gamma^\bullet_0 \circ \gamma^\bullet_1 : \mathrm{Cone}(f_{2,0}) \to \mathrm{Cone}(f_{0,0})$ satisfies the assumption of Proposition 2.33 since the composition of a compact morphism with other morphisms of Banach spaces is again a compact morphism, hence $R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(V, \Pi)$ is perfect.

Remark 4.17. It should be true that the existence of two resolutions $V_{r_0}^\bullet$ and $V_{r_1}^\bullet$ (instead of infinitely many) of V in Definition 4.1 is sufficient to deduce the same conclusion of Theorem 4.15. In fact, with a more careful argument, (4.7) can be represented by a strictly commutative diagram of bounded complexes of Banach spaces with vertical maps degreewise compact in the specific situation. Moreover, V is already prim with the two resolutions by the proof of Theorem 4.10. In all examples we know, representations of G with two resolutions $V_{r_k}^\bullet, k = 0, 1$ in Definition 4.1 are of compact presentation.

5. Dualities of supercuspidal representations of $\mathrm{GL}_2(\mathbb{Q}_p)$

In this section, we first recall some locally analytic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ which admits geometric realization using the Drinfeld tower. Then we compute the Schneider–Teitelbaum dual and the Bernstein–Zelevinsky dual of these representations. Finally, we give some applications by combining these two duality results: we discuss the resulting Serre-type duality and the trace pairing on the related extension groups.

Let $E/\mathbb{Q}_p$ be a sufficiently large finite extension and write $G = \mathrm{PGL}_2(\mathbb{Q}_p)$. Let $\mathbb{Q}_p^{\mathrm{ur}}$ be the maximal unramified extension of $\mathbb{Q}_p$ and let $\breve{\mathbb{Q}}_p$ be its completion. We normalize the local reciprocity map so that $p \in \mathbb{Q}_p^\times$ is sent to a geometric Frobenius element. Our convention is that the Hodge–Tate weight of the p-adic cyclotomic character $\epsilon : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathbb{Q}_p^\times$ is 1. If no further comment is made, locally analytic representations are solid. If V is a solid E-space, we write $V^\vee := \underline{\mathrm{Hom}}_E(V, E)$ for its (underived) E-linear dual and $V(*)$ for its underlying compactly generated topological space.

5.1. **The supercuspidal locally analytic representations.** We recall the definition of some locally analytic representations which can be constructed using the de Rham complex of the Drinfeld tower in [DLB17].

Let D_p be the unique quaternion division algebra over $\mathbb{Q}_p$, and let $D_p^\times$ be its group of units. Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_2(E)$ be a 2-dimensional de Rham representation, with Hodge–Tate weights $0, 1$ and $\det \rho \cong \epsilon$. We let M be the Fontaine's rank two $(\varphi, N, \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p))$-module $D_{\mathrm{pst}}(\rho \otimes \epsilon^{-1})$ over $E \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^{\mathrm{ur}}$ attached to ρ, and write $\mathrm{WD}(\rho) = \mathrm{WD}(M)$ for the Weil–Deligne representation associated to M (see [BS07, § 4]). We assume that $\mathrm{WD}(\rho)$ is absolutely irreducible.

Let $\Pi(\rho)$ be the admissible unitary Banach representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ attached to ρ under the p-adic local Langlands correspondence by Colmez [Col10], and write $\pi(\rho) := \Pi(\rho)^{la}$ for its subspace of locally analytic vectors; see also [CDN20, §0.2]. Our normalization follows [Col10, CDN20]: the central character of $\Pi(\rho)$ and $\pi(\rho)$ is $\det \rho \otimes \epsilon^{-1}$ which is trivial. Hence the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $\pi(\rho)$ factors through G.

The results of Dospinescu–Le Bras in [DLB17, Thm. 1.2 & Thm. 1.4] give a geometric realization of $\pi(\rho)$ using the de Rham complex of the Drinfeld tower. Let π_{sm} be the smooth supercuspidal representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ attached to $\mathrm{WD}(\rho)$ by the local Langlands correspondence, and let τ_{sm} be its Jacquet–Langlands transfer to $D_p^\times$. By our assumption, the central characters of π_{sm} and τ_{sm} are trivial.

Consider the Drinfeld tower $\{\Sigma_{m,E}\}_{m\in\mathbb{N}}$ in (3.16) of dimension 1 for $\mathbb{Q}_p$, see also [DLB17, § 3.1]. Define

$$\mathcal{O}[M] := \underline{\mathrm{Hom}}_{E[D_p^\times]}(\tau_{\mathrm{sm}}, \varinjlim_m H^0(\Sigma_{m,E}, \mathcal{O}_{\Sigma_{m,E}})),$$
$$\Omega^1[M] := \underline{\mathrm{Hom}}_{E[D_p^\times]}(\tau_{\mathrm{sm}}, \varinjlim_m H^0(\Sigma_{m,E}, \Omega^1_{\Sigma_{m,E}}))$$

which are Fréchet $\mathcal{D}^{la}(G, E)$-modules.

Definition 5.1. We define solid $\mathcal{D}^{la}(G, E)$-modules attached to M:

$$\pi_c := \mathcal{O}[M]^\vee, \quad \widetilde{\pi} := \Omega^1[M]^\vee, \tag{5.1}$$

which are admissible locally analytic representations by [DLB17, Thm. 1.2, Rem. 1.3(b)].

Put $M_{\mathrm{dR}} := (\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p^{\mathrm{ur}}} M)^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \simeq D_{\mathrm{dR}}(\rho \otimes \epsilon^{-1})$. This is a two-dimensional E-vector space. The de Rham differential induces a strict exact sequence

$$0 \to \mathcal{O}[M] \xrightarrow{d} \Omega^1[M] \to M_{\mathrm{dR}} \otimes_E \pi_{\mathrm{sm}}^\vee \to 0. \tag{5.2}$$

See [DLB17, Thm. 1.4] and [CDN20, § 2.1.5]. Taking the exact dual between Fréchet spaces and LS spaces ([RJRC22, Thm. 3.40]), one has a short exact sequence:

$$0 \longrightarrow \pi_{\mathrm{sm}}^{\oplus 2} \longrightarrow \widetilde{\pi} \longrightarrow \pi_c \longrightarrow 0 \tag{5.3}$$

of admissible locally analytic representations of G. The representation $\pi(\rho)$ is related to the exact sequence (5.3) as follows. We fix an isomorphism $D_{\mathrm{dR}}(\epsilon^{-1}) \simeq E$ so that we can identify $M_{\mathrm{dR}} \simeq D_{\mathrm{dR}}(\rho \otimes_E \epsilon^{-1}) \simeq D_{\mathrm{dR}}(\rho) \otimes_E D_{\mathrm{dR}}(\epsilon^{-1}) \simeq D_{\mathrm{dR}}(\rho)$.

Theorem 5.2 ([DLB17, Thm. 1.2 & Thm. 1.4])**.** *Let $\mathcal{L} = \mathrm{Fil}^0 D_{\mathrm{dR}}(\rho) \subset D_{\mathrm{dR}}(\rho)$ be the 1-dimensional Hodge filtration, viewed as an E-line of M_{dR} under the isomorphism $M_{\mathrm{dR}} \simeq D_{\mathrm{dR}}(\rho)$. Let $M_{\mathrm{dR}}^\vee \to \mathcal{L}^\vee$ be the quotient map. Then the following diagram is a pushout:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_{\mathrm{dR}}^\vee \otimes_E \pi_{\mathrm{sm}} & \longrightarrow & \widetilde{\pi} & \longrightarrow & \pi_c & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{L}^\vee \otimes_E \pi_{\mathrm{sm}} & \longrightarrow & \pi(\rho) & \longrightarrow & \pi_c & \longrightarrow & 0. \end{array} \tag{5.4}$$

Proof. For the E-line $\mathcal{L} \subset M_{\mathrm{dR}}$, let $W_{M,\mathcal{L}} \subset \Omega^1[M]$ be the inverse image of $\mathcal{L} \otimes_E \pi_{\mathrm{sm}}^\vee \subset M_{\mathrm{dR}} \otimes_E \pi_{\mathrm{sm}}^\vee$. By [CDN20, Prop. 2.9] (essentially [DLB17, Thm. 1.4]), there is a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant isomorphism $(W_{M,\mathcal{L}})^\vee \cong \pi(\rho)$. The result follows by taking E-duals. □

Hence $\pi(\rho)$ is an extension of π_c by π_{sm}. For later use, we also need to consider other such extensions, classified in [Din22].

Definition 5.3. For each E-line $\mathcal{L} \subset M_{\mathrm{dR}}$, we denote by $\pi_{\mathcal{L}}$ be the pushout of the map $M_{\mathrm{dR}}^\vee \otimes_E \pi_{\mathrm{sm}} \to \widetilde{\pi}$ along the map $M_{\mathrm{dR}}^\vee \otimes_E \pi_{\mathrm{sm}} \to \mathcal{L}^\vee \otimes_E \pi_{\mathrm{sm}}$ as in (5.4).

In particular, Theorem 5.2 gives $\pi(\rho) \cong \pi_{\mathcal{L}}$ with $\mathcal{L} = \mathrm{Fil}^0 D_{\mathrm{dR}}(\rho)$.

Theorem 5.4 ([Din22, Thm. 2.5])**.** *There is a natural isomorphism*

$$M_{\mathrm{dR}} \cong \mathrm{Ext}^1_{\mathcal{D}^{la}(G,E)}(\pi_c, \pi_{\mathrm{sm}})$$

such that an E-line $\mathcal{L} \subset M_{\mathrm{dR}}$ is sent to the class of the extension $\pi_{\mathcal{L}}$. Moreover, $\pi_{\mathcal{L}} \cong \pi_{\mathcal{L}'}$ if and only if $\mathcal{L} = \mathcal{L}'$.

Proof. The result in *loc. cit.* computed the extension group in the category of admissible $\mathrm{GL}_2(\mathbb{Q}_p)$-representations. Since there is no $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant map from π_c to π_{sm} by, for example, [Su25, Prop. 2.5.2], any such extension must have trivial central character. Hence the extension group coincides with the group of G-extensions, which also coincides with the extension group in the category of locally analytic representations of G over E by Proposition 5.6 below. For the last assertion, see for example [Su25, Prop. 2.6.3]. See also [Su25, Thm. 2.5.7 & Thm. 3.2.2 & Rem. 3.2.4] for more discussions. □

Remark 5.5. The representations above have analogues for arbitrary regular integral Hodge–Tate weights, using the corresponding equivariant vector bundles and dual BGG complexes ([DLB17, Rem. 1.3(a) and Rem. 11.12], [Su25, §2.2]), or using Colmez's construction ([Col19], [Din23, Thm. 1.1]). Similar discussions hold for these representations and the compatibility of the duality functors in this section with translation functors and fixing central characters is discussed in Appendix B, in particular Proposition B.9.

For later use, we also record here a compatibility result between the classical and solid extension groups of locally analytic representations. In particular, when discussing Hom or Ext^1 spaces, we can freely use results in classical settings and will omit further discussion of this point in the sequel.

Proposition 5.6. *Let V, W be admissible locally analytic representations. Taking the underlying topological spaces induces natural isomorphisms of E-vector spaces:*

$$\mathrm{Hom}_{\mathcal{D}^{la}(G,E)}(V, W) \xrightarrow{\sim} \mathrm{Hom}_G(V(*), W(*))$$
$$\mathrm{Ext}^1_{\mathcal{D}^{la}(G,E)}(V, W) \xrightarrow{\sim} \mathrm{Ext}^1_G(V(*), W(*))$$

where the groups on the right are computed in the abelian category of classical admissible locally analytic representations.

Proof. The map on Hom is an isomorphism by [RJRC22, Prop. 3.7] since the underlying topological spaces of admissible locally analytic representations are of compact type. Hence the abelian category of classical admissible locally analytic representations is embedded fully faithfully in $\mathrm{Rep}^{la}_{E_\square}(G)^\heartsuit$ and we write the category as $\mathrm{Rep}^{la}_{E_\square}(G)^{\heartsuit,\mathrm{adm}}$.

The Ext^1-groups in the abelian categories can be computed by Yoneda extensions [Sta25, Tag 06XU]. By the fully faithful embedding, it suffices to show if there is an extension

$$0 \to W \to T \to V \to 0$$

in $\mathrm{Rep}^{la}_{E_\square}(G)^\heartsuit$ such that if $W, V \in \mathrm{Rep}^{la}_{E_\square}(G)^{\heartsuit,\mathrm{adm}}$, then $T \in \mathrm{Rep}^{la}_{E_\square}(G)^{\heartsuit,\mathrm{adm}}$. Admissibility of G depends only on the restriction of T to some compact open subgroup $H \subset G$. Since V is of compact type, one has $R\underline{\mathrm{Hom}}_E(V, E) = V^\vee$ [RJRC22, Lem. 3.27]. Hence the dual sequence

$$0 \to V^\vee \to T^\vee \to W^\vee \to 0$$

is exact. If we can show that $T^\vee$ arises from a classical coadmissible $\mathcal{D}^{la}(H, E)(*)$-module, then taking the exact dual in [RJRC22, Thm. 3.40], we see $T^{\vee\vee}$ is an extension of V by W and we can conclude that $T \xrightarrow{\sim} T^{\vee\vee}$ by the five lemma. Then T is admissible.

To show $T^\vee$ is coadmissible, write $\mathcal{D}^{la}(H, E) = \varprojlim_n \mathcal{D}_n(H, E)$ as an inverse limit of Noetherian Banach E-algebras along (classically) flat and injective maps as in [ST03]. Finitely presented $\mathcal{D}_n(H, E)(*)$ are Banach modules with the canonical topology. It's then easy to see that the classical and solid finitely presented $\mathcal{D}_n(H, E)$-modules are equivalent and finitely presented modules in $\mathrm{Mod}_\square(\mathcal{D}_n(H, E))$ are stable under extensions. Moreover, by the flatness results in [ST03, Rem. 3.2] and arguments in [Bod26, Prop. 5.33] using Bar complexes, one can show that $\mathcal{D}_n(H, E) \otimes^{L\square}_{\mathcal{D}^{la}(H,E)} W^\vee \simeq \mathcal{D}_n(H, E) \otimes^{\square}_{\mathcal{D}^{la}(H,E)} W^\vee$ is finitely presented over $\mathcal{D}_n(H, E)$. We can then argue in the same way as in [Bre19, Lem. 2.1.1] to conclude that $T^\vee \simeq \varprojlim_n \mathcal{D}_n(H, E) \otimes^{\square}_{\mathcal{D}^{la}(H,E)} T^\vee$ and each $\mathcal{D}_n(H, E) \otimes^{\square}_{\mathcal{D}^{la}(H,E)} T^\vee$ is finitely presented. The result follows. □

5.2. **Schneider–Teitelbaum duality.** In this subsection, we compute the Schneider–Teitelbaum dual of $\pi(\rho)$ and representations $\pi_c, \widetilde{\pi}$ in Definition 5.1 (Proposition 5.12). The results below are known to experts but we can not find in the literature. And our methods seem to be new.

For $V \in \mathrm{Rep}^{la}_{E_\square}(G)$, the Schneider–Teitelbaum dual of V is defined as

$$\mathbb{D}_{\mathrm{ST}}(V) := R\underline{\mathrm{Hom}}(V, E)^{Rla}$$

where $(-)^{Rla}$ denotes the derived locally analytic vectors with respect to the action of G [RJRC22, Def. 4.40]; see [RJRC23, Prop. 4.2.7] for the compact case. We collect some basic properties of this functor in Appendix A, especially its relation with Schneider–Teitelbaum's original definition [ST05]: For a coadmissible $\mathcal{D}^{la}(G, E)$-module M, set

$$\mathbb{D}(M) := R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(M, \overline{\mathcal{D}}^{la}(G, E)[\dim G]),$$

see Definition A.5 for the notation and note that the character $\mathfrak{d}_G$ is trivial for out G by [ST05, Rem. 3.4]. Proposition A.6 gives an isomorphism

$$\mathbb{D}(V^\vee) \simeq R\underline{\mathrm{Hom}}_E(\mathbb{D}_{\mathrm{ST}}(V), E).$$

Remark 5.7. By the definition, the representation $\mathbb{D}_{\mathrm{ST}}(V)$ is independent of whether we consider V as a $\mathrm{PGL}_2(\mathbb{Q}_p)$-representation or a $\mathrm{GL}_2(\mathbb{Q}_p)$-representation.

We first determine the dual of some topologically irreducible representations. Let Δ be a (φ, Γ)-module of rank two over the Robba ring $\mathcal{R}_E$ for $\mathbb{Q}_p$ with the coefficients in E. We assume that Δ is indecomposable and let $\pi(\Delta)$ be the associated admissible locally analytic representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ in [Col16, Thm. 0.1]. Let $\Delta^\vee$ be the dual (φ, Γ)-module and write $\check{\Delta} := \Delta^\vee(\epsilon) = \Delta^\vee \otimes_{\mathcal{R}_E} D_{\mathrm{rig}}(\epsilon)$.

Proposition 5.8. *Suppose that $\pi(\Delta)(*)$ (resp. $\pi(\check{\Delta})(*)$) is topologically irreducible and $\pi(\Delta)^\vee(*)$ (resp. $\pi(\check{\Delta})^\vee(*)$) has (Gelfand–Kirillov) dimension* 1 *(*[ST03, § 8]*), then*

$$\mathbb{D}_{\mathrm{ST}}(\pi(\Delta)) \simeq \pi(\Delta^\vee(\epsilon))[-1]$$

in $\mathrm{Rep}^{la}_{E_\square}(\mathrm{GL}_2(\mathbb{Q}_p))$.

Proof. Let $H \subset \mathrm{GL}_2(\mathbb{Q}_p)$ be an open compact subgroup. The dimension 1 assumption on $\pi(\Delta)^\vee(*)$ implis that

$$\begin{aligned}\mathbb{D}(\pi(\Delta)^\vee(*)) &:= R\mathrm{Hom}_{\mathcal{D}^{la}(\mathrm{GL}_2(\mathbb{Q}_p),E)(*)}(\pi(\Delta)^\vee(*), \overline{\mathcal{D}}^{la}(\mathrm{GL}_2(\mathbb{Q}_p),E)(*))[4]\\ &\simeq R\mathrm{Hom}_{\mathcal{D}^{la}(H,E)(*)}(\pi(\Delta)^\vee(*), \overline{\mathcal{D}}^{la}(H,E)(*))[4]\end{aligned}$$

concentrates in degree -1 and 0. Since $\pi(\Delta)(*)$ has no dimension 0 quotient, it is pure in the sense of [ST03, Prop. 8.7]. By the argument of [HW24, Lem. 6.2] we find $\mathbb{D}(\pi(\Delta)^\vee(*))$ concentrates in degree -1. By [ST03, Lem. 8.4], [RJRC23, Rem. 4.2.8], $\mathbb{D}(\pi(\Delta)^\vee(*))$ or $\mathbb{D}(\pi(\Delta)^\vee)$ are both inverse limits of extension groups over Banach algebras. One can compare solid and classical Ext-groups between finitely presented modules over Noetherian Banach algebras as in [Bod26, § 5.4]. Consequently, we can deduce that $R\underline{\mathrm{Hom}}_E(\mathbb{D}_{\mathrm{ST}}(\pi(\Delta)), E) \simeq \mathbb{D}(\pi(\Delta)^\vee) \simeq \underline{\mathbb{D}(\pi(\Delta)^\vee)(*)}$ concentrates in a single degree. Since $\mathbb{D}_{\mathrm{ST}}(\pi(\Delta))$ is admissible in the sense of [RJRC23, Def. 4.2.1], $H^i(\mathbb{D}_{\mathrm{ST}}(\pi(\Delta)))$ are admissible, thus are acyclic for $R\underline{\mathrm{Hom}}_E(-,E)$[RJRC22, Lem. 3.27]. Then the spectral sequence for hypercohomology gives $H^i(R\underline{\mathrm{Hom}}_E(\mathbb{D}_{\mathrm{ST}}(\pi(\Delta)), E)) \simeq H^{-i}(\mathbb{D}_{\mathrm{ST}}(\pi(\Delta)))^\vee$. We deduce that $\mathbb{D}_{\mathrm{ST}}(\pi(\Delta))$ concentrates in degree 1. Similarly for $\mathbb{D}_{\mathrm{ST}}(\pi(\check{\Delta}))$.

We need to construct a non-zero morphism $\pi(\Delta) \to \mathbb{D}_{\mathrm{ST}}(\pi(\check{\Delta}))[1]$. Colmez constructed an extension

$$0 \to \pi(\check{\Delta})^\vee(*) \to \Delta \boxtimes_\omega \mathbb{P}^1 \to \pi(\Delta)(*) \to 0 \tag{5.5}$$

of topological $\mathcal{D}^{la}(\mathrm{GL}_2(\mathbb{Q}_p),E)(*)$-modules in [Col16, Thm. 0.1]. We claim that the extension (5.5) is non-split. Otherwise, since $\pi(\Delta)(*)$ is locally analytic, there exist non-zero locally analytic vectors in $\Delta\boxtimes_\omega\mathbb{P}^1$. But this is not possible, since one can check that the $\mathcal{D}^{la}(\begin{pmatrix}1 & \mathbb{Z}_p\\ 0 & 1\end{pmatrix}, E)$-module $\Delta \boxtimes \mathbb{Z}_p = \Delta \simeq \mathcal{R}_E^{\oplus 2}$ has no non-zero locally analytic vectors for the action of $\begin{pmatrix}1 & \mathbb{Z}_p\\ 0 & 1\end{pmatrix}$ (using for example [RJRC23, Thm. 4.1.5] and that $\mathcal{R}_E$ is the ring of functions on the boundary of the open unit disk).

By Colmez's construction $\Delta \boxtimes_\omega \mathbb{P}^1 \simeq \Delta \boxtimes \mathbb{Z}_p \oplus \Delta \boxtimes p\mathbb{Z}_p$ is an LF space since $\mathcal{R}_E$ is. Using [Bos23a, Lem. A.33] and [Ste24, Lem. 3.2], it is not hard to see that the condensification of (5.5) remains exact. The extension is also non-split by the same argument above, thus gives a non-zero element in

$$\begin{aligned}\mathrm{Ext}^1_{\mathcal{D}^{la}(\mathrm{GL}_2(\mathbb{Q}_p),E)}(\pi(\Delta), \pi(\check{\Delta})^\vee) &= \mathrm{Hom}_{\mathcal{D}^{la}(\mathrm{GL}_2(\mathbb{Q}_p),E)}(\pi(\Delta), \pi(\check{\Delta})^\vee[1])\\ &= \mathrm{Hom}_{\mathcal{D}^{la}(\mathrm{GL}_2(\mathbb{Q}_p),E)}(\pi(\Delta), \mathbb{D}_{\mathrm{ST}}(\pi(\check{\Delta}))[1])\end{aligned}$$

where the last identity is the adjunction in [RJRC23, Cor. 3.3.7]. Let $f : \pi(\Delta) \to \mathbb{D}_{\mathrm{ST}}(\pi(\check{\Delta}))[1]$ be the non-zero morphism given by the extension. Then f is an injection since both the source and target are admissible (hence f is classical) and $\pi(\Delta)(*)$ is topologically irreducible. Applying $\mathbb{D}_{\mathrm{ST}}(-)[1]$, the non-zero map $\mathbb{D}_{\mathrm{ST}}(f)[1] : \pi(\check{\Delta}) \to \mathbb{D}_{\mathrm{ST}}(\pi(\Delta))[1]$ is also injective by the same reason, which implies that f is a surjection. The result follows. □

Remark 5.9. Proposition 5.8 applies for all indecomposable Δ unless Δ is trianguline or Δ is de Rham with regular Hodge–Tate weights up to twist [Col19, Thm.0.3] (the dimension 1 assumption is always satisfied by [DPS23, Thm. 1.5]). If Δ is trianguline, $\mathbb{D}_{\mathrm{ST}}(\pi(\Delta))$ can be computed using [ST05, § 6]. The result for the proposition also holds for regular de Rham Δ by Proposition 5.12 below.

Remark 5.10. Proposition 5.8 says that $\pi(\Delta)$ is self-dual up to a twist under $\mathbb{D}_{\mathrm{ST}}(-)$. When the representations appear in the completed cohomology or its patched versions, such self-dual result can often be obtained using the Poincaré duality of the completed cohomology or the big patching module, see for example [HW22, Thm. 8.2] and [Din24, Appendix D].

Lemma 5.11. *We have* $\mathbb{D}_{\mathrm{ST}}(\pi_{\mathrm{sm}}) \simeq \pi_{\mathrm{sm}}^{\vee,\mathrm{sm}}[0]$ *where* $\pi_{\mathrm{sm}}^{\vee,\mathrm{sm}} \simeq \pi_{\mathrm{sm}}$ *is the smooth contragredient of* π_{sm}.

Proof. By [RJRC23, Prop. 5.3.2], $\pi_{\mathrm{sm}}^{\vee,R\mathrm{sm}} \simeq \pi_{\mathrm{sm}}^{\vee,Rla}$ where $(-)^{R\mathrm{sm}}$ is defined for $\mathcal{D}^{\mathrm{sm}}(G,E)$-modules in *loc.cit.*. Since $(-)^{\mathrm{sm}}$ is exact [RJRC23, Lem. 5.1.9], $\pi_{\mathrm{sm}}^{\vee,Rla} = \pi_{\mathrm{sm}}^{\vee,\mathrm{sm}}$. Since π_{sm} has trivial central character, its contragredient is π_{sm} itself, by [BH06, § 33.4, Prop. (1) & Exercise]. □

Next, we determine $\mathbb{D}_{\mathrm{ST}}(\pi_c), \mathbb{D}_{\mathrm{ST}}(\widetilde{\pi})$ using the results in [Din23]. Let $D = D_{\mathrm{rig}}(\rho)$ be the (φ, Γ)-module over $\mathcal{R}_E$ associated to the Galois representation ρ in the last subsection. Let Δ be the p-adic differential equation associated to $D_{\mathrm{pst}}(\rho)$. In other words, Δ is the unique (φ, Γ)-submodule of Hodge–Tate weights 0 of $D_{\mathrm{rig}}(\rho)[\frac{1}{t}]$ given by [Din23, Prop. 2.11], the change of weights of $D_{\mathrm{rig}}(\rho)$. Let $\lambda = (0, 0)$ and $\mu = (-1, 0)$ considered as the weights of the diagonal Cartan of $\mathfrak{gl}_2$. By [Dos12], the infinitesimal characters of the Harish-Chandra center of $U(\mathfrak{gl}_2)$ of $\pi(\rho), \pi(\Delta)$ are given by that of the Verma modules (with respect to the Borel subalgebra of the upper-triangular matrices) of the highest weights λ and μ respectively. Let $T^{\lambda}_{\mu}, T^{\mu}_{\lambda}$ be the translation functor recalled in § B. By [Din23, Thm. 1.1(2)] (see also [Su25, Prop. 2.4.2]), there exists a short exact sequence

$$0 \to \pi_c \to T^{\lambda}_{\mu}\pi(\Delta) \to \widetilde{\pi} \to 0 \tag{5.6}$$

of locally analytic representations. Moreover, one has $T^{\mu}_{\lambda}\widetilde{\pi} \simeq T^{\mu}_{\lambda}\pi_c \simeq \pi(\Delta)$.

Proposition 5.12. *We have*

$$\mathbb{D}_{\mathrm{ST}}(\pi_c) \simeq \widetilde{\pi}[-1], \qquad \mathbb{D}_{\mathrm{ST}}(\widetilde{\pi}) \simeq \pi_c[-1],$$

and

$$\mathbb{D}_{\mathrm{ST}}(\pi(\rho)) \simeq \pi(\rho)[-1].$$

Proof. A similar proof of [LSW26, Thm. 3.11], along with (5.3), shows that $\pi_c, \pi(\Delta)$ as well as $\widetilde{\pi}$ have Gelfand–Kirillov dimension 1. Since $\pi_c(*), \pi(\Delta)(*)$ are topologically irreducible by [Col19, Cor. 0.4] and the sequence (5.3) is the universal extension of π_c by π_{sm} in Theorem 5.4, all these representations are pure. Hence by the arguments in the proof of Proposition 5.8, $\mathbb{D}_{\mathrm{ST}}(\pi_c), \mathbb{D}_{\mathrm{ST}}(\widetilde{\pi}), \mathbb{D}_{\mathrm{ST}}(\pi(\Delta))$ concentrates in degree 1. Moreover, applying $\mathbb{D}_{\mathrm{ST}}$ to (5.3) and using Lemma 5.11 give a short exact sequence

$$0 \to \pi_{\mathrm{sm}}^{\oplus 2} \to \mathbb{D}_{\mathrm{ST}}(\pi_c)[1] \to \mathbb{D}_{\mathrm{ST}}(\widetilde{\pi})[1] \to 0. \tag{5.7}$$

On the other hand, by Proposition 5.8, we have $\mathbb{D}_{\mathrm{ST}}(\pi(\Delta) \simeq \pi(\Delta^{\vee}(\epsilon))[-1]$. Since $\det \Delta = t^{-1}\mathcal{R}(\epsilon)$, $\Delta^{\vee}(\epsilon) \simeq t\Delta$ so that $\pi(\Delta^{\vee}(\epsilon)) \simeq \pi(\Delta) \otimes \det$. By [RJRC23, Cor. 3.2.14(3)] or using Proposition B.7, $\mathbb{D}_{\mathrm{ST}}$ is compatible with the translation functors. Hence applying $\mathbb{D}_{\mathrm{ST}}$ to (5.6) gives another short exact sequence

$$0 \to \mathbb{D}_{\mathrm{ST}}(\widetilde{\pi})[1] \to T^{\lambda}_{\mu}\pi(\Delta) \to \mathbb{D}_{\mathrm{ST}}(\pi_c)[1] \to 0.$$

Combined with (5.7) considering the irreducible constituents of $T^{\lambda}_{\mu}\pi(\Delta)(*)$, one has $\mathbb{D}_{\mathrm{ST}}(\widetilde{\pi})[1] \simeq \pi_c$. Hence $\mathbb{D}_{\mathrm{ST}}(\pi_c) \simeq \mathbb{D}_{\mathrm{ST}}(\mathbb{D}_{\mathrm{ST}}(\widetilde{\pi})[1]) \simeq \widetilde{\pi}[-1]$.

Note that since $\det \rho \simeq \epsilon$, we have $\rho^{\vee} \otimes \epsilon \simeq \rho$. With the determination of $\mathbb{D}_{\mathrm{ST}}(\pi_c)$ and the methods in Proposition 5.8, the last isomorphism for $\pi(\rho)$, which is a non-split extension of π_c by π_{sm}, is an easy exercise. We leave it to the reader. □

5.3. **Bernstein–Zelevinsky duality.** In this subsection, we compute the Bernstein–Zelevinsky dual (3.1) of $\pi(\rho)$ and some relevant locally analytic representations of $G = \mathrm{PGL}_2(\mathbb{Q}_p)$, using Theorem 3.16.

Remark 5.13. Unlike the Schneider–Teitelbaum duality, $\mathbb{D}_{\mathrm{BZ}}(\pi(\rho))$ depends on whether we consider $\pi(\rho)$ as a G-representation or a $\mathrm{GL}_2(\mathbb{Q}_p)$-representation (differed by a degree shift), see Remark 5.7 and Remark 3.23.

Lemma 5.14. *There exists an isomorphism* $\mathbb{D}_{\mathrm{BZ}}(\pi_{\mathrm{sm}}) \simeq \pi_{\mathrm{sm}}[0]$.

Proof. By [SW25, Rem. 5.5], $\mathbb{D}_{\mathrm{BZ}}(\pi_{\mathrm{sm}})$ coincides with the classical Bernstein–Zelevinsky dual of π_{sm}. The result now follows from [SS97, Rem. III.3.4] and the self-duality of π_{sm} for the smooth contragredient in Lemma 5.11. □

We keep the notation in § 5.1. for the de Rham Galois representation ρ and the Weil–Deligne representation M. We associated a smooth representation τ_{sm} of $D_p^{\times}$. As in Lemma 5.11, since $\rho \simeq \rho^{\vee} \otimes \epsilon$, one has $\tau_{\mathrm{sm}}^{\vee} \simeq \tau_{\mathrm{sm}}$ (cf. [BH06, § 56.1, (56.1.4)]). Let

$$\mathcal{F}_M := \mathcal{F}_{\tau_{\mathrm{sm}}} \simeq \mathcal{F}_{\tau_{\mathrm{sm}}^{\vee}} \simeq \mathcal{F}_M^{\vee}$$

be the G-equivariant vector bundle on $X := \Omega_{\mathrm{Dr},E} = (\mathbb{P}^{1,\mathrm{an}}_{\mathbb{Q}_p} \setminus \mathbb{P}^1(\mathbb{Q}_p))_E$ attached to τ_{sm} by Definition 3.21 and Lemma 3.22. Let $\Omega^1_X = \mathcal{O}_X(-2) := \mathcal{O}_{\mathbb{P}^{1,\mathrm{an}}_E}(-2)|_X$ be the dualizing line bundle on

X. By Serre duality, étale descent, projection formula, and Definition 5.1, we have

$$\begin{aligned} \widetilde{\pi} &\simeq \Gamma(X, \mathcal{F}_M^\vee \otimes_{\mathcal{O}_X} \Omega_X^1)^\vee \simeq H_c^1(X, \mathcal{F}_M) \\ \pi_c &\simeq \Gamma(X, \mathcal{F}_M^\vee)^\vee \simeq H_c^1(X, \mathcal{F}_M \otimes_{\mathcal{O}_X} \Omega_X^1). \end{aligned} \tag{5.8}$$

Moreover, by the discussion around (5.6), together with the results on the translation of equivariant sheaves in [Su25, § 2], we have

$$\pi(\Delta) \simeq \Gamma(X, \mathcal{F}_M^\vee(-1))^\vee \simeq H_c^1(X, \mathcal{F}_M(-1)), \tag{5.9}$$

where (-1) denotes the twist by the tautological line bundle on $\mathbb{P}_E^{1,\mathrm{an}}$.

Proposition 5.15. *We have isomorphisms*

$$\theta_{\widetilde{\pi}} : \mathbb{D}_{\mathrm{BZ}}(\widetilde{\pi}) \simeq \pi_c[-1], \qquad \theta_{\pi_c} : \mathbb{D}_{\mathrm{BZ}}(\pi_c) \simeq \widetilde{\pi}[-1],$$

and

$$\theta_{T_\lambda^\mu \pi(\Delta)} : \mathbb{D}_{\mathrm{BZ}}(T_\lambda^\mu \pi(\Delta)) \simeq T_\lambda^\mu \pi(\Delta)[-1]$$

in $\mathrm{Rep}_{E_\square}^{la}(G)$.

Proof. The first two isomorphisms follow from Corollary 3.19, (5.8), (5.9) and the self-duality of $\mathcal{F}_M$. The last is by the compatibility between the translation functor and $\pi(\Delta) = T_\lambda^\mu \pi_c$. □

Let $d : \mathcal{O}_{\Sigma_{m,E}} \to \Omega^1_{\Sigma_{m,E}}$ be the differential map (see (3.16) for $\Sigma_{m,E}$) and also write $d : \mathcal{F}_M \to \mathcal{F}_M \otimes_{\mathcal{O}_X} \Omega_X^1$ for the induced map by descent.

Lemma 5.16. *The following diagram commute:*

$$\begin{array}{ccc} \widetilde{\pi} & \xrightarrow{\quad -d \quad} & \pi_c \\ {\scriptstyle \theta_{\widetilde{\pi}}}\downarrow{\scriptstyle \sim} & & {\scriptstyle \theta_{\pi_c}}\downarrow{\scriptstyle \sim} \\ \mathbb{D}_{\mathrm{BZ}}(\pi_c)[1] & \xrightarrow{\mathbb{D}_{\mathrm{BZ}}(d)[1]} & \mathbb{D}_{\mathrm{BZ}}(\widetilde{\pi})[1]. \end{array} \tag{5.10}$$

Proof. Applying the Serre duality, a calculation using the residue pairing in [RJRC22, Lem. 2.16] shows that the map $\mathbb{D}_{\mathrm{GS}}(d)[-1] : \mathcal{F}_M \to \mathcal{F}_M \otimes_{\mathcal{O}_X} \Omega_X^1$ coincides with $-d$. Let $U \subset X$ be an affinoid, the following diagram commutes:

$$\begin{array}{ccc} H^0(U, \mathcal{F}_M) & \xrightarrow{\quad d \quad} & H^0(U, \mathcal{F}_M \otimes_{\mathcal{O}_X} \Omega_X^1) \\ \downarrow{\scriptstyle \sim} & & {\scriptstyle \sim}\downarrow \\ H_c^1(U, \mathcal{F}_M \otimes_{\mathcal{O}_X} \Omega_X^1)^\vee & \xrightarrow{\quad -d^\vee \quad} & H_c^1(U, \mathcal{F}_M)^\vee. \end{array}$$

Now the commutativity of (5.10) follows from the proof of Theorem 3.16 using resolutions by compactly induced representations in Proposition 3.15. □

We define

$$R\Gamma_{\mathrm{dR},c}(X, \mathcal{F}_M) := R\Gamma_c(X, [\mathcal{F}_M \xrightarrow{d} \mathcal{F}_M \otimes_{\mathcal{O}_X} \Omega_X^1]) \tag{5.11}$$

where the compactly supported cohomology is taken in the classical sense (cf. § 2.3). Then the dual of (5.2) induces an isomorphism

$$H_{\mathrm{dR},c}^1(X, \mathcal{F}_M) = \ker(\widetilde{\pi} \xrightarrow{d} \pi_c) \simeq M_{\mathrm{dR}}^\vee \otimes_E \pi_{\mathrm{sm}}.$$

Applying $\mathbb{D}_{\mathrm{BZ}}$, we obtain an isomorphism

$$\begin{aligned} H_{\mathrm{dR},c}^1(X, \mathcal{F}_M) \simeq \mathrm{fib}(\widetilde{\pi} \to \pi_c) \xrightarrow{(5.10)} \mathrm{fib}(\mathbb{D}_{\mathrm{BZ}}(\pi_c) \to \mathbb{D}_{\mathrm{BZ}}(\widetilde{\pi}))[1] &\simeq \mathbb{D}_{\mathrm{BZ}}(\mathrm{fib}(\widetilde{\pi} \to \pi_c)) \\ &\simeq \mathbb{D}_{\mathrm{BZ}}(H_{\mathrm{dR},c}^1(X, \mathcal{F}_M)) \end{aligned} \tag{5.12}$$

which is uniquely defined up to scalars. By the definition of $\mathbb{D}_{\mathrm{GS}}$ and tensor-Hom adjunction (and the $E_\square$-flatness of the LS spaces), the above isomorphism induces a pairing

$$H_{\mathrm{dR},c}^1(X, \mathcal{F}_M) \otimes_{E_\square} H_{\mathrm{dR},c}^1(X, \mathcal{F}_M) \to \mathcal{C}_c^{la}(G, E) \tag{5.13}$$

equivariant for the actions of $G \times G$ on the two sides. Since

$$\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G\times G, E)}(\pi_{\mathrm{sm}} \otimes_{E_\square} \pi_{\mathrm{sm}}, \mathcal{C}_c^{la}(G, E)) \simeq \underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G, E)}(\pi_{\mathrm{sm}}, \mathbb{D}_{\mathrm{BZ}}(\pi_{\mathrm{sm}})) \simeq E,$$

the map (5.13) induces a pairing

$$(-,-) : M_{\mathrm{dR}}^{\vee} \otimes_{E_\square} M_{\mathrm{dR}}^{\vee} \to \mathrm{Hom}_{\mathcal{D}^{la}(G\times G,E)}(\pi_{\mathrm{sm}} \otimes_{E_\square} \pi_{\mathrm{sm}}, \mathcal{C}_c^{la}(G,E)) \simeq E \tag{5.14}$$

which is perfect.

Proposition 5.17. *The pairing* (5.14) *is alternating:* $(x,y) = -(y,x)$ *for all* $x, y \in M_{\mathrm{dR}}^{\vee}$.

Proof. Suppose that there is a map $\alpha : \pi_1 \to \mathbb{D}_{\mathrm{BZ}}(\pi_2)$, which induces by adjunction a map

$$\pi_1 \otimes^L_{E_\square} \pi_2 \to \mathcal{C}_c^{la}(G,E).$$

Since $\otimes_{E_\square}$ is symmetric, there is a natural isomorphism $\mathrm{sw} : \pi_2 \otimes^L_{E_\square} \pi_1 \simeq \pi_1 \otimes^L_{E_\square} \pi_2$. There is also an involution $\iota : \mathcal{C}_c^{la}(G,E) \to \mathcal{C}_c^{la}(G,E)$ induced $g \mapsto g^{-1}$. By the tensor Hom adjunction we obtain a map $\alpha^t : \pi_2 \to \mathbb{D}_{\mathrm{BZ}}(\pi_1)$ which is the transpose of α with respect $\mathbb{D}_{\mathrm{BZ}}$. Take $\pi_1 = \mathbb{D}_{\mathrm{BZ}}(\pi_2)$ this gives the canonical map $\eta_{\pi_2} : \pi_2 \to \mathbb{D}_{\mathrm{BZ}}(\mathbb{D}_{\mathrm{BZ}}(\pi_2))$. In general $\alpha^t = \mathbb{D}_{\mathrm{BZ}}(\alpha) \circ \eta_{\pi_2}$. It's not hard to show that (5.14) is alternating if and only if $\alpha^t = -\alpha$ where α denotes the map in (5.12).

We need to be extremely careful about the sign. Write $\epsilon_{\pi,s} : \mathbb{D}_{\mathrm{BZ}}(\pi)[-s] \xrightarrow{\sim} \mathbb{D}_{\mathrm{BZ}}(\pi[s])$ for the canonical isomorphism. Under the convention of [Sta25, Tag 0FNG, (9)], $\epsilon_{\pi,s}$ sends an element of degree m in the Hom complex by $(-1)^{ms}$. Hence we have

$$\epsilon_{\pi[b],a} \circ (\epsilon_{\pi,b}[-a]) = (-1)^{ab}\epsilon_{\pi,a+b}. \tag{5.15}$$

For $\alpha : \pi_1 \to \mathbb{D}_{\mathrm{BZ}}(\pi_2)$ and $s \in \mathbb{Z}$, we have

$$(\epsilon_{\pi_2,-s} \circ (\alpha[s]))^t = (-1)^s \epsilon_{\pi_1,s} \circ (\alpha^t[-s]). \tag{5.16}$$

This is because the following diagram

$$\begin{array}{ccc} \pi_2[-s] \otimes^L_{E_\square} \pi_1[s] & \xrightarrow{\mathrm{sw}} & \pi_1[s] \otimes^L_{E_\square} \pi_2[-s] \\ \sim\downarrow & & \downarrow\sim \\ \pi_2 \otimes^L_{E_\square} \pi_1 & \xrightarrow{\mathrm{sw}} & \pi_1 \otimes^L_{E_\square} \pi_2. \end{array}$$

commutes up to $(-1)^{s^2} = (-1)^s$ under the convention of [Sta25, Tag 0FNG].

We now argue as in [Bal00, Thm. 1.6]. Consider the following commutative diagram

(5.17)

$$\begin{array}{ccccccc} \widetilde{\pi} & \xrightarrow{d} & \pi_c & \xrightarrow{u_1} & H^1_{\mathrm{dR},c}(X,\mathcal{F}_M)[1] & \xrightarrow{u_2} & \widetilde{\pi}[1] \\ {\scriptstyle\theta_{\widetilde{\pi}}}\downarrow\sim & & {\scriptstyle\theta_{\pi_c}}\downarrow\sim & & \psi\downarrow\sim & & {\scriptstyle\theta_{\widetilde{\pi}}[1]}\downarrow\sim \\ \mathbb{D}_{\mathrm{BZ}}(\pi_c)[1] & \xrightarrow{-\mathbb{D}_{\mathrm{BZ}}(d)[1]} & \mathbb{D}_{\mathrm{BZ}}(\widetilde{\pi})[1] & \xrightarrow{\mathbb{D}_{\mathrm{BZ}}(u_2)[2]} & \mathbb{D}_{\mathrm{BZ}}(H^1_{\mathrm{dR},c}(X,\mathcal{F}_M))[1] & \xrightarrow{\mathbb{D}_{\mathrm{BZ}}(u_1)[2]} & \mathbb{D}_{\mathrm{BZ}}(\pi_c)[2]. \end{array}$$

The bottom is obtained from the top triangle by applying $\mathbb{D}_{\mathrm{BZ}}(-)$ and rotating. The left square commutes by (5.10). The existence of ψ follows from the axiom TR 3 of the triangulated category. In our case, the map ψ is uniquely defined by the above diagram using that the map $\mathbb{D}_{\mathrm{BZ}}(u_1)[1]$ is injective. By our definition, $\psi[-1]$ coincides with (5.12). Thus we only need to show that $\psi[-1]^t = -\psi[-1]$.

Applying $\mathbb{D}_{\mathrm{BZ}}(-)$ to (5.17) and using $\eta_{\pi_c[-2]}$, $\eta_{H^1_{\mathrm{dR},c}(X,\mathcal{F}_M)[-1]}$, $\eta_{\widetilde{\pi}[-1]}$ and $\eta_{\pi_c[-1]}$, we obtain the following transpose of (5.17):

(5.18)

$$\begin{array}{ccccccc} \pi_c[-2] & \xrightarrow{u_1[-2]} & H^1_{\mathrm{dR},c}(X,\mathcal{F}_M)[-1] & \xrightarrow{u_2[-2]} & \widetilde{\pi}[-1] & \xrightarrow{-d[-1]} & \pi_c[-1] \\ {\scriptstyle(\theta_{\widetilde{\pi}}[1])^t}\downarrow\sim & & \psi^t\downarrow\sim & & \theta^t_{\pi_c}\downarrow\sim & & \theta^t_{\widetilde{\pi}}\downarrow\sim \\ \mathbb{D}_{\mathrm{BZ}}(\widetilde{\pi})[-1] & \xrightarrow{\mathbb{D}_{\mathrm{BZ}}(u_2)} & \mathbb{D}_{\mathrm{BZ}}(H^1_{\mathrm{dR},c}(X,\mathcal{F}_M))[-1] & \xrightarrow{\mathbb{D}_{\mathrm{BZ}}(u_1)} & \mathbb{D}_{\mathrm{BZ}}(\pi_c) & \xrightarrow{\mathbb{D}_{\mathrm{BZ}}(d)} & \mathbb{D}_{\mathrm{BZ}}(\widetilde{\pi}). \end{array}$$

Here we omitted the isomorphisms $\epsilon_{\pi,s}$. We have an identity $(\theta_{\pi_c}[-1])^t = \epsilon_{\pi_c,-1} \circ \theta_{\widetilde{\pi}} : \widetilde{\pi} \to \mathbb{D}_{\mathrm{BZ}}(\pi_c[-1]) \simeq \mathbb{D}_{\mathrm{BZ}}(\pi_c)[1]$. To see this, we need know the isomorphism $\theta_{\pi_c}[-1] : \pi_c[-1] \simeq \mathbb{D}_{\mathrm{BZ}}(\widetilde{\pi})$ and the pairing $\widetilde{\pi} \otimes^L_{E_\square} \pi_c[-1] \to \mathcal{C}_c^{la}(G,E)$ from Proposition 5.15 give the natural pairing $\widetilde{\pi} \otimes^L_{E_\square} \mathbb{D}_{\mathrm{BZ}}(\widetilde{\pi}) \to \mathcal{C}_c^{la}(G,E)$. Taking a resolution of $\widetilde{\pi}$ in (1) of Proposition 3.15, this follows from the proof of Theorem 3.16.

Applying (5.16) and using (5.15), we get

$$\begin{aligned}\theta^t_{\pi_c} &= -\epsilon_{\pi_c[-1],1}\circ((\theta_{\pi_c}[-1])^t[-1])\\ &= -\epsilon_{\pi_c[-1],1}\circ(\epsilon_{\pi_c,-1}[-1])\circ(\theta_{\widetilde{\pi}}[-1])\\ &= \theta_{\widetilde{\pi}}[-1].\end{aligned}$$

Similarly, we have $\theta^t_{\widetilde{\pi}} = \theta_{\pi_c}[-1]$. By the commutativity of the second square of (5.18) and the right square of (5.17), we have

$$\begin{aligned}\mathbb{D}_{\mathrm{BZ}}(u_1)[1]\circ\psi^t[1] &= \theta_{\widetilde{\pi}}\circ u_2[-1],\\ \mathbb{D}_{\mathrm{BZ}}(u_1)[1]\circ\psi[-1] &= \theta_{\widetilde{\pi}}\circ u_2[-1].\end{aligned} \tag{5.19}$$

Since $\mathbb{D}_{\mathrm{BZ}}(u_1)[1]$ is injective, we conclude that $\psi^t[1]=\psi[-1]$. Applying (5.16), we get $(\psi[-1])^t = -\psi^t[1] = -\psi[-1]$, as desired. □

Theorem 5.18. *We have*

$$\mathbb{D}_{\mathrm{BZ}}(\pi(\rho)) \simeq \pi(\rho)[-1].$$

Proof. Let $\mathcal{L}\subset M_{\mathrm{dR}}$ and let $\pi_{\mathcal{L}}$ be the G-representation constructed in Definition 5.3. By Lemma 5.14, Proposition 5.15, applying $\mathbb{D}_{\mathrm{BZ}}$ to (5.4), we get a diagram:

(5.20)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{L}\otimes_E\pi_{\mathrm{sm}} & \xrightarrow{\ i\ } & \widetilde{\pi} & \longrightarrow & \mathbb{D}_{\mathrm{BZ}}(\pi_{\mathcal{L}})[1] & \longrightarrow & 0\\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & M_{\mathrm{dR}}\otimes_E\pi_{\mathrm{sm}}\simeq\mathbb{D}_{\mathrm{BZ}}(M_{\mathrm{dR}}^\vee\otimes_E\pi_{\mathrm{sm}}) & \longrightarrow & \widetilde{\pi}=\mathbb{D}_{\mathrm{BZ}}(\pi_c)[1] & \longrightarrow & \pi_c\simeq\mathbb{D}_{\mathrm{BZ}}(\widetilde{\pi})[1] & \longrightarrow & 0.
\end{array}$$

It follows that $\mathbb{D}_{\mathrm{BZ}}(\pi_{\mathcal{L}})[1]\simeq\pi_{\mathcal{L}'}$ for some E-line $\mathcal{L}'\subset M_{\mathrm{dR}}$ by Theorem 5.4. The remaining task is to show $\mathcal{L}'=\mathcal{L}$. Notice that the second row of the above diagram is obtained by applying $\mathbb{D}_{\mathrm{BZ}}$ to the universal extension and corresponds to the map $M_{\mathrm{dR}}\simeq\mathrm{Ext}^1_{\mathcal{D}^{la}(G,E)}(\pi_c,\pi_{\mathrm{sm}})\to\mathrm{Hom}_{\mathcal{D}^{la}(G,E)}(\pi_{\mathrm{sm}},\widetilde{\pi})\simeq M_{\mathrm{dR}}^\vee$, or a pairing

$$(-,-): M_{\mathrm{dR}}^\vee\times M_{\mathrm{dR}}^\vee\to E. \tag{5.21}$$

The desired identity $\mathcal{L}'=\mathcal{L}$ is equivalent to that the composition $\mathcal{L}\hookrightarrow M_{\mathrm{dR}}\to M_{\mathrm{dR}}^\vee\to\mathcal{L}^\vee$ is zero. This is equivalent to that $(x,x)=0$ for all $x\in\mathcal{L}$. Let $\mathcal{L}$ vary, we need prove that the pairing (5.21) is alternating. Since (5.12) induced by $\mathbb{D}_{\mathrm{BZ}}$ also sends $H^1_{\mathrm{dR},c}(X,\mathcal{F}_M)=M_{\mathrm{dR}}^\vee\otimes_E\pi_{\mathrm{sm}}\twoheadrightarrow\mathcal{L}^\vee\otimes_E\pi_{\mathrm{sm}}$ to the map $\mathcal{L}\otimes_E\pi_{\mathrm{sm}}\hookrightarrow\mathbb{D}_{\mathrm{BZ}}(H^1_{\mathrm{dR},c}(X,\mathcal{F}_M))[1]=\mathrm{fib}(\mathbb{D}_{\mathrm{BZ}}(\pi_c)\to\mathbb{D}_{\mathrm{BZ}}(\widetilde{\pi}))[1]$ in (5.20), the pairing (5.21) coincides with (5.14) and we conclude by Proposition 5.17. □

Remark 5.19. There is another proof for Theorem 5.18 using results in § 5.2. For $V\in\mathrm{Rep}^{la}_{E_\square}(G)$, the inclusion $\mathcal{C}^{la}_c(G,E)\hookrightarrow\mathcal{C}^{la}(G,E)=(\mathrm{Ind}^G_{\{1\}}E)^{la}$ induces a map

$$\eta_V:\mathbb{D}_{\mathrm{BZ}}(V)^{Rla}\to\mathbb{D}_{\mathrm{ST}}(V)$$

via

$$R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(V,\mathcal{C}^{la}(G,E))\simeq R\underline{\mathrm{Hom}}_E(V,E).$$

If V is smooth and supercuspidal, one can check that η_V is an isomorphism. Using Proposition 5.12, we get a diagram

$$\begin{array}{ccccccc}
\pi_{\mathrm{sm}} & \longrightarrow & \widetilde{\pi} & \longrightarrow & \mathbb{D}_{\mathrm{BZ}}(\pi(\rho))[1] & \longrightarrow & \pi_{\mathrm{sm}}[1]\\
\downarrow{\scriptstyle\eta_{\pi_{\mathrm{sm}}}} & & \downarrow{\scriptstyle\eta_{\pi_c}[1]} & & \downarrow{\scriptstyle\eta_{\pi(\rho)}[1]} & & \downarrow{\scriptstyle\eta_{\pi_{\mathrm{sm}}}[1]}\\
\pi_{\mathrm{sm}} & \longrightarrow & \widetilde{\pi} & \longrightarrow & \pi(\rho) & \longrightarrow & \pi_{\mathrm{sm}}[1].
\end{array}$$

Then the first vertical arrow is an isomorphism. This forces the second vertical arrow non-zero. But $\mathrm{End}_{\mathcal{D}^{la}(G,E)}(\widetilde{\pi})=E$ by [Su25, Prop. 2.5.2 & Prop. 2.6.1 & Rem. 2.5.9], we see η_{π_c} is an isomorphism. We conclude that $\eta_{\pi(\rho)}$ is also an isomorphism.

5.4. **Serre functor and trace pairings.** In this subsection, we combine the Schneider–Teitelbaum and Bernstein–Zelevinsky duality to establish some Serre-type duality on extension groups of locally analytic representations in § 5.1.

Let $\langle \pi_{\mathrm{sm}}, \pi_c \rangle \subset \mathrm{Rep}^{la}_{E_\square}(G)$ be the full triangulated subcategory generated by $\pi_{\mathrm{sm}}, \widetilde{\pi}, \pi_c$ in the sense of [Sta25, Tag 0ATG]. Objects in $\langle \pi_{\mathrm{sm}}, \pi_c \rangle$ can be obtained from π_{sm}, π_c via finitely many operations consisting of cones/extensions, shifts and retracts. In particular, $\pi(\rho) \in \langle \pi_{\mathrm{sm}}, \pi_c \rangle$.

We first collect the following finiteness result.

Proposition 5.20. *For any $\pi_1, \pi_2 \in \langle \pi_{\mathrm{sm}}, \pi_c \rangle$, the object $R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(\pi_1, \pi_2)$ is perfect in $D(E_\square)$.*

Proof. By [BH06, §15.5], $\pi_{\mathrm{sm}} \cong \text{c-Ind}_J^G \sigma$ for a compact open subgroup $J \subset G$ and a finite-dimensional representation σ of J. This implies that π_{sm} is of compact presentation; see also Example 4.3. By Proposition 4.7 and (5.8), π_c is of compact presentation. Moreover, π_{sm}, π_c are admissible. Hence $R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(\pi_1, \pi_2)$ is perfect if $\pi_1, \pi_2 \in \{\pi_c, \pi_{\mathrm{sm}}\}$ by Theorem 4.15. The result now follows by dévissage. □

Proposition 5.21. *Let $\pi \in \langle \pi_{\mathrm{sm}}, \pi_c \rangle$ or π is of compact presentation, and let $V \in \mathrm{Rep}^{la}_{E_\square}(G)$ such that the underlying solid E-vector space of V is nuclear* [Bos23b, Def. A.10]*. There is a natural equivalence*

$$\mathbb{D}_{\mathrm{BZ}}(\pi) \otimes^{L\square}_{\mathcal{D}^{la}(G,E)} V[-3] \simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(\pi, V).$$

Proof. We first construct a natural map between the two sides of the desired isomorphism. Let $K \subset G$ be an open compact subgroup. By [RJRC23, Prop. 3.2.12(2)& Prop. A.2.1(4)], one has (see Lemma A.2 and Proposition A.4 for the G-equivariance):

$$\begin{aligned} \mathcal{C}^{la}_c(G,E) \otimes^{L\square}_{\mathcal{D}^{la}(G,E)} V[-3] &\simeq \mathcal{C}^{la}_c(K,E) \otimes^{L\square}_{\mathcal{D}^{la}(K,E)} V[-3] \\ &\simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(K,E)}(\mathcal{D}^{la}(K,E), V) \\ &\simeq V. \end{aligned}$$

For any $\pi \in \mathrm{Rep}^{la}_{E_\square}(G)$, the natural map

$$\pi \otimes^{L\square}_E \mathbb{D}_{\mathrm{BZ}}(\pi) \otimes^{L\square}_{\mathcal{D}^{la}(G,E)} V[-3] \to \mathcal{C}^{la}_c(G,E) \otimes^{L\square}_{\mathcal{D}^{la}(G,E)} V[-3] \simeq V$$

is $\mathcal{D}^{la}(G,E)$-equivariant, by tensor-Hom adjunction, inducing a natural map

$$f_\pi : \mathbb{D}_{\mathrm{BZ}}(\pi) \otimes^{L\square}_{\mathcal{D}^{la}(G,E)} V[-3] \to R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(\pi, V).$$

Since π_c, π_{sm} are of compact presentation, by dévissage, it suffices to prove that f_π is an equivalence when $\pi = \text{c-Ind}_K^G W$ where $K \subset G$ is compact open and W is a locally analytic K-representation whose underlying solid E-space is Smith. By [RJRC23, Prop. A.2.1(4)],

$$R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(\pi, V) \simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(K,E)}(W, V) \simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(K,E)}(E, R\underline{\mathrm{Hom}}_E(W, V)).$$

Since V is nuclear and W is Smith, the evaluation map $W^\vee \otimes^L_{E_\square} V \to R\underline{\mathrm{Hom}}_E(W, V)$ is an equivalence by [And21, Prop. 5.35] and [RJRC22, Lem. 3.8(2) & Lem. 3.10]. On the other hand, Proposition 3.2 gives

$$\mathbb{D}_{\mathrm{BZ}}(\pi) \simeq W^\vee \otimes^{L\square}_{\mathcal{D}^{la}(K,E)} \mathcal{D}^{la}(G,E).$$

By [RJRC23, Prop. 3.2.12(2)] again, we get

$$R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(K,E)}(E, W^\vee \otimes^L_{E_\square} V) \simeq (W^\vee \otimes^{L\square}_{\mathcal{D}^{la}(K,E)} V)[-3].$$

This gives the desired equivalence for f_π. □

Remark 5.22. The same proof as above shows that for a general p-adic Lie group G, $\pi, V \in \mathrm{Rep}^{la}_{E_\square}(G)$ with $\pi \in \mathrm{Rep}^{la}_{E_\square}(G)^\omega$ and with the underlying E-space of V nuclear, we have a natural isomorphism

$$(\mathbb{D}_{\mathrm{BZ}}(\pi) \otimes \mathfrak{d}^*_G) \otimes^{L\square}_{\mathcal{D}^{la}(G,E)} V[-\dim G] \simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(\pi, V).$$

where $\mathfrak{d}^*_G$ is the right character of G in § A. Moreover, by [HM24, Lem. 4.4.6], the above isomorphism holds for any prim π and arbitrary V, cf. Theorem 4.10.

Definition 5.23. We define the "Serre functor" $\mathcal{S} : \mathrm{Rep}^{la}_{E_\square}(G) \to \mathrm{Mod}_\square(\mathcal{D}^{la}(G,E))$ by

$$V \mapsto \mathcal{S}(V) := \mathbb{D}_{\mathrm{ST}}(\mathbb{D}_{\mathrm{BZ}}(V))[\dim G].$$

Corollary 5.24. *For $\pi_1, \pi_2 \in \langle \pi_{\mathrm{sm}}, \pi_c \rangle$, there is a natural equivalence of perfect complexes in $D(E_\square)$:*

$$R\underline{\mathrm{Hom}}_E(R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(\pi_1,\pi_2),E) \simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(\pi_2, \mathcal{S}(\pi_1)).$$

Proof. The representations π_{sm} and π_c are admissible, in particular they are nuclear by [Bos23b, Prop. A.11] being LS of compact type. Hence any object of $\langle \pi_{\mathrm{sm}}, \pi_c \rangle$ is nuclear by [And21, Lem. 5.40] and is admissible in the sense of [RJRC23, Def. 4.2.1]. Then one has

$$\begin{aligned} R\underline{\mathrm{Hom}}_E(R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(\pi_1,\pi_2),E) &\simeq R\underline{\mathrm{Hom}}_E(\mathbb{D}_{\mathrm{BZ}}(\pi_1) \otimes^{L\square}_{\mathcal{D}^{la}(G,E)} \pi_2, E)[3] \\ &\simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(\mathbb{D}_{\mathrm{BZ}}(\pi_1), R\underline{\mathrm{Hom}}_E(\pi_2,E))[3] \\ &\simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(\mathbb{D}_{\mathrm{BZ}}(\pi_1), \mathbb{D}_{\mathrm{ST}}(\pi_2))[3] \\ &\simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(\pi_2, \mathbb{D}_{\mathrm{ST}}(\mathbb{D}_{\mathrm{BZ}}(\pi_1))[3]) \\ &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(\pi_2, \mathcal{S}(\pi_1)). \end{aligned}$$

where the first isomorphism is Proposition 5.21, the third one is the adjunction in [RJRC23, Cor. 3.3.7], and the fourth is by Lemma A.7. □

We now define trace pairings on extension groups using Corollary 5.24. For each $V \in \{\pi_{\mathrm{sm}}, \widetilde{\pi}, \pi_c, \pi(\rho)\}$, results in § 5.2 and § 5.3 give an isomorphism $\mathbb{D}_{\mathrm{ST}}(\mathbb{D}_{\mathrm{BZ}}(V)) \simeq V$ and we fix one for each V. Let

$$\Phi_V : R\mathrm{Hom}_{\mathcal{D}^{la}(G,E)}(V, V[3]) \xrightarrow{\sim} R\mathrm{Hom}_E(R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(V,V),E)$$

be the inverse of the equivalence in Corollary 5.24, after taking the underlying complex of abstract E-vector spaces. Define the trace map associated to V by

$$\mathrm{tr}_V : \mathrm{Ext}^3_{\mathcal{D}^{la}(G,E)}(V,V) \longrightarrow E, \qquad x \longmapsto H^0(\Phi_V)(x)(\mathrm{id}_V).$$

For $V, W \in \{\pi_{\mathrm{sm}}, \widetilde{\pi}, \pi_c, \pi(\rho)\}$ and $i \in \mathbb{Z}$, the *trace pairing* is

$$\mathrm{Ext}^i_{\mathcal{D}^{la}(G,E)}(V,W) \times \mathrm{Ext}^{3-i}_{\mathcal{D}^{la}(G,E)}(W,V) \longrightarrow E, \qquad (f,g) \longmapsto \mathrm{tr}_V(g \circ f),$$

where $g \circ f$ denotes the Yoneda product.

Proposition 5.25. *The trace pairing above is perfect for every $V, W \in \{\pi_{\mathrm{sm}}, \widetilde{\pi}, \pi_c, \pi(\rho)\}$ and $i \in \mathbb{Z}$.*

Proof. Fix $V, W \in \{\pi_{\mathrm{sm}}, \widetilde{\pi}, \pi_c, \pi(\rho)\}$. By Corollary 5.24, we obtain a natural equivalence

$$\Phi_{V,W} : R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(W, V[3]) \xrightarrow{\sim} R\underline{\mathrm{Hom}}_E(R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(V,W),E).$$

Taking cohomology in degree $-i$ and the underlying abstract E-spaces (using Proposition 5.20) give an isomorphism

$$\mathrm{Ext}^{3-i}_{\mathcal{D}^{la}(G,E)}(W,V) \xrightarrow{\sim} \mathrm{Hom}_E(\mathrm{Ext}^i_{\mathcal{D}^{la}(G,E)}(V,W),E).$$

We check that this isomorphism is induced by the trace pairing. For $f : V \to W[i]$ and $g : W \to V[3-i]$, naturality of $\Phi_{V,W}$ gives a commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}(W[i], V[3]) & \xrightarrow{H^0(\Phi_{V,W[i]})} & \mathrm{Hom}_E(\mathrm{Hom}(V, W[i]), E) \\ \Big\downarrow{\scriptstyle -\circ f} & & \Big\downarrow{\scriptstyle (f\circ -)^\vee} \\ \mathrm{Hom}(V, V[3]) & \xrightarrow{H^0(\Phi_V)} & \mathrm{Hom}_E(\mathrm{Hom}(V,V),E). \end{array}$$

Hence

$$H^0(\Phi_{V,W[i]})(g[i])(f) = H^0(\Phi_V)(g \circ f)(\mathrm{id}_V) = \mathrm{tr}_V(g \circ f).$$

Thus the isomorphism $H^0(\Phi_{V,W[i]})$ sends $g[i]$ to $f \mapsto \mathrm{tr}_V(g \circ f)$. Since $H^0(\Phi_{V,W[i]})$ is an isomorphism between finite-dimensional vector spaces, the trace pairing is perfect. □

Appendix A. Schneider–Teitelbaum duality for locally analytic representations

In this appendix we collect some results on Schneider–Teitelbaum duality [ST05] for solid locally analytic representations, using the machinery developed in [RJRC22, RJRC23].

Let E be a finite extension of $\mathbb{Q}_p$. Let G be a p-adic Lie group of dimension d. Similarly as $\mathcal{C}^{la}(G,E), \mathcal{C}_c^{la}(G,E), \mathcal{D}^{la}(G,E)$, we write $\mathcal{C}^{\mathrm{sm}}(G,E)$, $\mathcal{C}_c^{\mathrm{sm}}(G,E)$, and $\mathcal{D}^{\mathrm{sm}}(G,E)$ for the space of smooth functions, the space of compactly supported smooth functions, and the smooth distribution algebra in [RJRC23, § 5], respectively. Let

$$\overline{\mathcal{D}}^{la}(G,E) := \underline{\mathrm{Hom}}_{E_\square}(\mathcal{C}_c^{la}(G,E), E)$$

be the space of locally analytic distributions on G, with its natural $\mathcal{D}^{la}(G,E)$-bimodule structure. We work in $D_\square(\mathcal{D}^{la}(G,E))$, the stable ∞-category of solid $\mathcal{D}^{la}(G,E)$-modules.

Let $\mathfrak{g} = E \otimes_{\mathbb{Q}_p} \mathrm{Lie}\, G$ and $\Delta_G = \bigwedge_E^d \mathfrak{g}$, with the adjoint G-action. Let δ_G be the smooth modulus character, normalized by $\mu_G(gXg^{-1}) = \delta_G(g)\mu_G(X)$ for compact open subsets $X \subset G$, where μ_G is a fixed E-valued left Haar measure on G, and put $\mathfrak{d}_G = \Delta_G \otimes_E \delta_G$. For $\chi \in \{\Delta_G, \delta_G, \mathfrak{d}_G\}$, we view χ as a left representation and write χ^* for its E-linear dual, regarded as a right representation by $(\lambda \cdot g)(v) = \lambda(gv)$. Through the involution $g \mapsto g^{-1}$, χ^* becomes the usual left representation χ^{-1}, which is the inverse of χ.

Remark A.1. The right character $\mathfrak{d}_G^*$ coincides with the right character with the same notation in [ST05, Prop. 3.5]. Note that in [ST05, p. 301], δ_G denotes the same modulus character but G acts non-trivially on the right.

In the next statements, $(-)^{Rla}$ on $\mathcal{D}^{la}(G,E)$ is taken for the left regular action.

Lemma A.2. *For $M \in D_\square(\mathcal{D}^{la}(G,E))$, the natural morphism*

$$\mathcal{D}^{la}(G,E)^{Rla} \otimes^L_{\mathcal{D}^{la}(G,E)} M \longrightarrow M^{Rla}$$

is an equivalence.

Proof. This is essentially a reformulation of [RJRC23, Cor. 3.2.14(1)]. The natural map $\mathcal{D}^{la}(G,E)^{Rla} \to \mathcal{D}^{la}(G,E)$ induces a morphism $\mathcal{D}^{la}(G,E)^{Rla} \otimes^L_{\mathcal{D}^{la}(G,E)} M \to \mathcal{D}^{la}(G,E) \otimes^L_{\mathcal{D}^{la}(G,E)} M \simeq M$. Its source is derived locally analytic, so by [RJRC23, Cor. 3.3.7] this morphism factors through M^{Rla}. By [RJRC23, Prop. 3.2.6(3)], both sides of the morphism preserve colimits. It is therefore enough to take $M = \mathcal{D}^{la}(G,E) \otimes_{E_\square} E_\square[S], S \in \mathrm{ExtDisc}$ to check the equivalence, which follows from [RJRC23, Cor. 3.2.15]. □

Lemma A.3. *There exists a natural isomorphism*

$$R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(E, \overline{\mathcal{D}}^{la}(G,E)) \simeq \mathfrak{d}_G^*[-d]$$

of right $\mathcal{D}^{la}(G,E)$-modules.

Proof. If G is compact, this is [RJRC23, Lem. 3.2.10]. For non-compact G, this is a consequence of [ST05, Cor. 3.7] whose proof can be adapted to the solid setting. □

We prove the following proposition, which is [RJRC23, Cor. 3.2.14(1)] when G is compact, in order to identify the Schneider–Teitelbaum duality for locally analytic representations with the duality for solid $\mathcal{D}^{la}(G,E)$-modules. See Proposition A.6 and [RJRC23, Rem. 1.2.3(1)].

Proposition A.4. *There is a natural equivalence of $\mathcal{D}^{la}(G,E)$-bimodules*

$$\mathcal{D}^{la}(G,E)^{Rla} \simeq \mathcal{C}_c^{la}(G,E) \otimes_{E_\square} \mathfrak{d}_G^*[-d].$$

Put $l_{g_1} f(g) = f(g_1^{-1}g)$ and $r_{g_2} f(g) = f(gg_2^{-1})$. Then the regular actions of g on $\mathcal{D}^{la}(G,E)$ correspond to the actions

$$g \cdot (f \otimes \lambda) = l_g f \otimes \lambda, \qquad (f \otimes \lambda) \cdot g = r_g f \otimes (\lambda \cdot g),$$

on the right, where $g \in G$, $f \in \mathcal{C}_c^{la}(G,E)$, and $\lambda \in \mathfrak{d}_G^*$.

Proof. Let $K \subset G$ be an open compact subgroup. By [RJRC23, Cor. 3.2.15], there is an isomorphism of $\mathcal{D}^{la}(K,E)$-bimodules

$$\mathcal{D}^{la}(K,E)^{Rla} \simeq \mathcal{C}^{la}(K,E) \otimes_{E_\square} \Delta_G^*[-d].$$

Under the above isomorphism, the actions on $\mathcal{C}^{la}(K,E)$ are given by l_g and r_g, by its proof using [RJRC23, Cor. 3.2.14 & Prop. A.2.1(3)]. Applying c-Ind$_K^G$ we obtain a left $\mathcal{D}^{la}(G,E)$-linear equivalence

$$\mathcal{D}^{la}(G,E)^{Rla} \simeq \mathcal{C}_c^{la}(G,E) \otimes_{E_\square} \Delta_G^*[-d], \tag{A.1}$$

which is also right $\mathcal{D}^{la}(K,E)$-linear. Then it suffices to identify the right $\mathcal{D}^{la}(G,E)$-action on the right-hand side.

Consider the space $\mathcal{C}^{la}(G,\mathcal{D}^{la}(G,E))$ with G-actions. If $f: G \to \mathcal{D}^{la}(G,E)$ and $g_i \in G_i = G, i = 1,2,3,4$, then $(g_1,g_2,g_3,g_4).f(-) = g_3 f(g_1^{-1} - g_2)g_4$. By [RJRC23, Def. 3.2.3(3)], $\mathcal{D}^{la}(G,E)^{Rla} \simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(E, \mathcal{C}^{la}(G,\mathcal{D}^{la}(G,E))_{*_{1,3}})$ where $*_{1,3}$ denotes the diagonal action of $G \subset G_1 \times G_3$.

If $K \subset G$ is compact open, one has $\mathcal{C}^{la}(K,\mathcal{D}^{la}(G,E)) \simeq \mathcal{C}^{la}(K,E) \otimes_{E_\square} \mathcal{D}^{la}(G,E)$. For $g \in G$, let δ_g be the Dirac measure. There is a twisting isomorphism

$$\mathcal{C}^{la}(K,E)_0 \otimes_{E_\square} \delta_g \mathcal{D}^{la}(K,E) \simeq \mathcal{C}^{la}(gK,E) \otimes_{E_\square} \delta_g \mathcal{D}^{la}(K,E), f \otimes \delta_g \mu \mapsto \sum_i \delta_g \mu_{1,i}.f \otimes \delta_g \mu_{2,i}$$

if $\nabla(\mu) = \sum_i \mu_{1,i} \otimes \mu_{2,i}$ with the inverse $f \otimes \delta_g \mu \mapsto \sum_i S(\mu_{1,i})\delta_g^{-1} f \otimes \delta_g \mu_{2,i}$ (∇ is the comultiplication, and S is the antipode of $\mathcal{D}^{la}(K,E)$, cf. [RJRC23, Prop. A.2.1(3)]). Here $\mathcal{C}^{la}(K,E)_0$ denotes the same space of $\mathcal{C}^{la}(K,E)$ with the trivial left actions of G. We obtain a map

$$\mathcal{C}^{la}(K,E)_0 \otimes_{E_\square} \delta_g \mathcal{D}^{la}(K,E) \to \mathcal{C}^{la}(gK, \mathcal{D}^{la}(G,E))$$

for each $g \in G/K$. Since $\mathcal{C}^{la}(G,\mathcal{D}^{la}(G,E)) = \prod_{g \in G/K} \mathcal{C}^{la}(gK,\mathcal{D}^{la}(G,E))$ and $\overline{\mathcal{D}}^{la}(G,E) = \prod_{g\in G/K} \delta_g \mathcal{D}^{la}(K,E)$, taking direct product, we obtain a morphism

$$\mathcal{C}^{la}(K,E)_0 \otimes_{E_\square} \overline{\mathcal{D}}^{la}(G,E) \to \mathcal{C}^{la}(G,\mathcal{D}^{la}(G,E)).$$

The map is K-equivariant for the right translations on $\mathcal{C}^{la}(K,E)_0$ and the action of $K \subset G_2$ on the target. By the Frobenius reciprocity, we obtain the following map

$$\Phi : \mathcal{C}_c^{la}(G,E)_0 \otimes_{E_\square} \overline{\mathcal{D}}^{la}(G,E) \to \mathcal{C}^{la}(G,\mathcal{D}^{la}(G,E)), \tag{A.2}$$

which extends the more transparent map where $\overline{\mathcal{D}}^{la}(G,E)$ is replaced by $\mathcal{D}^{la}(G,E)$. Informally, for $h \in \mathcal{C}^{la}(G,E)$ and $x \in G$, one has $\Phi(f\otimes\mu)(x)(h) = \mu(y \mapsto f(y^{-1}x)h(y))$. The map Φ is equivariant for the following actions: $g_2, g_4 \in G_2 \times G_4$ acts on the left hand side by $f\otimes\mu \mapsto f(g_4 - g_2)\otimes\mu\delta_{g_4}$, and the $*_{1,3}$-action on the right corresponds to the left action on the $\overline{\mathcal{D}}^{la}(G,E)$-factor. Taking derived G-invariants of (A.2) for the last actions and using Lemma A.3, we get a $G_2 \times G_4$-equivariant map

$$\mathcal{C}_c^{la}(G,E) \otimes_{E_\square} \mathfrak{d}_G^*[-d] \to \mathcal{D}^{la}(G,E)^{Rla}. \tag{A.3}$$

Using the isomorphism $f \to (g \mapsto f(g^{-1}))$ of $\mathcal{C}_c^{la}(G,E)$ we recover the desired actions of G on the left hand side. Comparing with (A.1) and restricting (A.3) to $\mathcal{C}_c^{la}(K,E) \otimes_{E_\square} \mathfrak{d}_K^*[-d] \to \mathcal{D}^{la}(K,E)^{Rla}$ we see (A.3) is an isomorphism. □

For a compact open subgroup $K \subset G$, let $\mathcal{D}^h(K,E) = \mathcal{D}^{(h^+)}(K,E)$ be the h-analytic distribution algebras and $\mathrm{Mod}^{\mathrm{qc}}_{E_\square}(\mathcal{D}^{la}(K,E))$ the category of quasi-coherent modules over $\mathcal{D}^{la}(K,E)$ of [RJRC23, Def. 4.1.1]. The objects of $\mathrm{Mod}^{\mathrm{qc}}_{E_\square}(\mathcal{D}^{la}(K,E))$ are compatible systems $(M_h)_h$ of solid $\mathcal{D}^h(K,E)$-modules in $D_\square(\mathcal{D}^h(K,E))$. There is a localization functor:

$$D_\square(\mathcal{D}^{la}(K,E)) \to \mathrm{Mod}^{\mathrm{qc}}_{E_\square}(\mathcal{D}^{la}(K,E)), \quad M \mapsto j^*M = (\mathcal{D}^h(K,E) \otimes^{L\square}_{\mathcal{D}^{la}(K,E)} M)_h,$$

which admits a fully faithful right adjoint $(M_h)_h \mapsto j_*(M_h)_h = R\varprojlim_h M_h$ and a fully faithful left adjoint $(M_h)_h \mapsto j_!(M_h)_h = (R\varprojlim_h M_h)^{Rla}$ [RJRC23, Lem. 4.1.4 & Thm. 4.1.5]. We write $\mathrm{Rep}^{\mathrm{ad}}_{E_\square}(G) \subset \mathrm{Rep}^{la}_{E_\square}(G)$ for the full subcategory consisting of admissible representations, whose restriction to K is admissible in the sense of [RJRC23, Def. 4.2.1]. Let $\mathbb{D}_{\mathrm{qc}}$ denote the duality on $\mathrm{Mod}^{\mathrm{qc}}_{E_\square}(\mathcal{D}^{la}(K,E))$ of [RJRC23, Def. 4.1.11].

Definition A.5. For $V \in D_\square(\mathcal{D}^{la}(G,E))$, let $V^\vee := R\underline{\mathrm{Hom}}_{E_\square}(V,E)$ with the usual dual action. For $V \in \mathrm{Rep}^{la}_{E_\square}(G)$, put

$$\mathbb{D}_{\mathrm{ST}}(V) := (V^\vee)^{Rla}.$$

For $M \in D_{\square}(\mathcal{D}^{la}(G,E))$, put

$$\mathbb{D}(M) := R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(M, \overline{\mathcal{D}}^{la}(G,E) \otimes_{E_{\square}} \mathfrak{d}_G[d]).$$

Here $\overline{\mathcal{D}}^{la}(G,E) \otimes_{E_{\square}} \mathfrak{d}_G[d] \simeq (\mathcal{C}_c^{la}(G,E) \otimes_{E_{\square}} \mathfrak{d}_G^*[-d])^{\vee}$ is equipped with the (left and right) dual actions of G in Proposition A.4 and $\mathbb{D}(M)$ is a left $\mathcal{D}^{la}(G,E)$-module induced by the right action on $\overline{\mathcal{D}}^{la}(G,E) \otimes_{E_{\square}} \mathfrak{d}_G[d]$ and the involution $g \mapsto g^{-1}$ of G.

Proposition A.6. *For every solid $\mathcal{D}^{la}(G,E)$-module M, there is a natural equivalence of solid $\mathcal{D}^{la}(G,E)$-modules*

$$(M^{Rla})^{\vee} \simeq \mathbb{D}(M).$$

In particular, $\mathbb{D}_{\mathrm{ST}}(V)^{\vee} \simeq \mathbb{D}(V^{\vee})$ for $V \in \mathrm{Rep}^{la}_{E_{\square}}(G)$.

Proof. Lemma A.2, tensor-Hom adjunction, and Proposition A.4 give

$$\begin{aligned}(M^{Rla})^{\vee} &\simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(M, (\mathcal{D}^{la}(G,E)^{Rla})^{\vee}) \\ &\simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(M, \overline{\mathcal{D}}^{la}(G,E) \otimes_{E_{\square}} \mathfrak{d}_G[d]),\end{aligned}$$

which is G-equivariant. □

We now collect some adjunction and biduality properties of the Schneider–Teitelbaum duality functor $\mathbb{D}_{\mathrm{ST}}$.

Lemma A.7. *For $V, W \in \mathrm{Rep}^{la}_{E_{\square}}(G)$, there is a natural equivalence in $D(E_{\square})$:*

$$R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(V, \mathbb{D}_{\mathrm{ST}}(W)) \simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(W, \mathbb{D}_{\mathrm{ST}}(V)).$$

Proof. The adjunction of $(-)^{Rla}$ in [RJRC23, Cor. 3.3.7] and tensor-Hom adjunction give

$$\begin{aligned}R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(V, \mathbb{D}_{\mathrm{ST}}(W)) &\simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(V, W^{\vee}) \\ &\simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(W, V^{\vee}) \\ &\simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(W, \mathbb{D}_{\mathrm{ST}}(V)).\end{aligned}$$

Here, the second equivalence follows because [RJRC23, Prop. A.2.1(2),(4)] identifies both sides with $R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(V \otimes^L_{E_{\square}} W, E) \simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(E, R\underline{\mathrm{Hom}}_E(V \otimes^L_{E_{\square}} W, E))$. □

For $V \in \mathrm{Rep}^{la}_{E_{\square}}(G)$, the canonical map $V \to V^{\vee\vee}$ together with $\mathbb{D}_{\mathrm{ST}}(V) \to V^{\vee}$ induces a map $V \longrightarrow V^{\vee\vee} \longrightarrow \mathbb{D}_{\mathrm{ST}}(V)^{\vee}$ which factors through a canonical map

$$V \longrightarrow \mathbb{D}_{\mathrm{ST}}(\mathbb{D}_{\mathrm{ST}}(V)) =: \mathbb{D}^2_{\mathrm{ST}}(V)$$

by the adjunction of $(-)^{Rla}$.

Proposition A.8. *For $V \in \mathrm{Rep}^{\mathrm{ad}}_{E_{\square}}(G)$, the representation $\mathbb{D}_{\mathrm{ST}}(V)$ is admissible and the natural morphism $V \to \mathbb{D}^2_{\mathrm{ST}}(V)$ is an equivalence.*

Proof. It suffices to prove the result after restricting to an open compact subgroup $K \subset G$. Let $j_!, j_*, j^*$ and $\mathbb{D}_{\mathrm{qc}}$ be the functors in [RJRC23, § 4] recalled above. Since V is admissible, $(M_h)_h := j^*V$ is coherent in the sense of [RJRC23, Def. 4.2.1] and $V = j_!j^*V$. Each M_h is perfect, so $\mathbb{D}_{\mathrm{qc}}((M_h)_h) = (R\underline{\mathrm{Hom}}_{\mathcal{D}^h(K,E)}(M_h, \mathcal{D}^h(K,E) \otimes_{E_{\square}} \mathfrak{d}_G[d]))_h$ is coherent and the biduality for perfect objects gives $\mathbb{D}^2_{\mathrm{qc}}(j^*V) \simeq j^*V$. By [RJRC23, Prop. 4.2.7], we have $\mathbb{D}_{\mathrm{ST}}(V) \simeq j_!\mathbb{D}_{\mathrm{qc}}j^*V$, which is admissible. Using the same proposition and the full faithfulness of $j_!$, we get

$$\begin{aligned}\mathbb{D}^2_{\mathrm{ST}}(V) &\simeq j_!\mathbb{D}_{\mathrm{qc}}j^*j_!\mathbb{D}_{\mathrm{qc}}j^*V \\ &\simeq j_!\mathbb{D}^2_{\mathrm{qc}}j^*V \\ &\simeq j_!j^*V = V.\end{aligned}$$

The result follows. □

Corollary A.9. *The functor $\mathbb{D}_{\mathrm{ST}}$ restricts to an anti-equivalence*

$$\mathbb{D}_{\mathrm{ST}} : (\mathrm{Rep}^{\mathrm{ad}}_{E_{\square}}(G))^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Rep}^{\mathrm{ad}}_{E_{\square}}(G),$$

whose quasi-inverse is $\mathbb{D}_{\mathrm{ST}}$. Let $V \in \mathrm{Rep}^{la}_{E_{\square}}(G)$ and $W \in \mathrm{Rep}^{\mathrm{ad}}_{E_{\square}}(G)$. Applying $\mathbb{D}_{\mathrm{ST}}$ induces a natural equivalence in $D(E_{\square})$

$$R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(V, W) \simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(\mathbb{D}_{\mathrm{ST}}(W), \mathbb{D}_{\mathrm{ST}}(V)).$$

Proof. Proposition A.8 shows that $\mathbb{D}_{\mathrm{ST}}$ preserves admissibility and that the natural morphism $\mathrm{id} \to \mathbb{D}_{\mathrm{ST}}^2$ is an equivalence. The isomorphism for $R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(-,-)$ follows from Lemma A.7 and $W \simeq \mathbb{D}_{\mathrm{ST}}^2(W)$. □

Appendix B. Translation functors and duality functors

In this appendix, we give a solid treatment of translation functors for locally analytic representations, extending the usual formalism in [JLS24]. We discuss their compatibility with Schneider–Teitelbaum and Bernstein–Zelevinsky duality. We will also give similar statements for representations with fixed central character.

We start with the general framework for translation functors. Let L be a finite extension of $\mathbb{Q}_p$. Let $\mathbb{G}$ be a connected split reductive group over L, put $G = \mathbb{G}(L)$ and see it as a locally $\mathbb{Q}_p$-analytic group, and fix a split maximal torus $T = \mathbb{T}(L)$ contained in a fixed Borel subgroup B of $\mathbb{G}$. Take E large enough such that $\mathrm{Hom}_{\mathbb{Q}_p}(L, E) = [L : \mathbb{Q}_p]$ and write $\mathfrak{g} = E \otimes_{\mathbb{Q}_p} \mathrm{Lie}\, G$ and $\mathfrak{z} = Z(U(\mathfrak{g}))$, the Harish–Chandra center of the universal enveloping algebra. The image of $\mathfrak{z}$ is central in $\mathcal{D}^{la}(G, E)$ [ST02a, Prop. 3.7]. For $V \in D_\square(\mathcal{D}^{la}(G, E))$ and $\mathfrak{m} \in \mathrm{MaxSpec}(\mathfrak{z})$ (the spectrum of maximal ideals of $\mathfrak{z}$), put

$$V_{\mathfrak{m}} := \mathrm{colim}_{n \geq 1} R\underline{\mathrm{Hom}}_{\mathfrak{z}}(\mathfrak{z}/\mathfrak{m}^n, V).$$

Then we get a natural map $V_{\mathfrak{m}} \to V$. Write $\mathrm{pr}_{\mathfrak{m}}(V) := V_{\mathfrak{m}}$.

Definition B.1. We say that V is $\mathfrak{z}$-finite if there is a finite subset $\Sigma \subset \mathrm{MaxSpec}(\mathfrak{z})$ for which the natural morphism

$$\bigoplus_{\mathfrak{m} \in \Sigma} V_{\mathfrak{m}} \longrightarrow V$$

is an equivalence and, for every $\mathfrak{m} \in \Sigma$, there exists a positive integer N such that $\mathfrak{m}^N$ acts by zero on $V_{\mathfrak{m}}$, i.e. $V_{\mathfrak{m}}$ is in the image $D_\square(\mathcal{D}^{la}(G, E)/\mathfrak{m}^N) \to D_\square(\mathcal{D}^{la}(G, E))$ via the forgetful functor.

Remark B.2. One can show that $\mathcal{D}^{la}(G, E)/\mathfrak{m}^N \simeq \mathcal{D}^{la}(G, E) \otimes_{\mathfrak{z}}^{L\square} \mathfrak{z}/\mathfrak{m}^N$ using Kostant's theorem that $U(\mathfrak{g})$ is free over $\mathfrak{z}$ and the arguments for [SS16, Prop. 4.8].

Let ρ be the half-sum of the positive roots. For a weight $\lambda \in (E \otimes_{\mathbb{Q}_p} \mathrm{Lie}\, T)^*$, write $|\lambda|$ for the orbit of λ under the dot action $w \cdot \lambda = w(\lambda + \rho) - \rho$. Let $\chi_\lambda : \mathfrak{z} \to E$ be the infinitesimal character of the Verma module of highest weight λ. Put $\mathfrak{m}_\lambda = \ker(\chi_\lambda)$ which depends only on $|\lambda|$. Write $\mathrm{pr}_{|\lambda|} = \mathrm{pr}_{\mathfrak{m}_\lambda}$.

Definition B.3. We say that an object $V \in D_\square(\mathcal{D}^{la}(G, E))$ lies in the $|\lambda|$-block if the natural map $\mathrm{pr}_{|\lambda|}(V) \to V$ is an equivalence.

Lemma B.4. *Suppose that $V \in D_\square(\mathcal{D}^{la}(G, E)/\mathfrak{m}_\lambda^N)$ for some λ and $N \geq 1$. Then V lies in the $|\lambda|$-block.*

Proof. By the Harish–Chandra isomorphism, $\mathfrak{z}$ is Noetherian. Since $\mathfrak{m}_\lambda^N$ acts by zero on V, all its cohomology modules are $\mathfrak{m}_\lambda$-torsion. Thus the natural map $\mathrm{pr}_{|\lambda|}(V) \to V$ is an equivalence by the arguments for [Sta25, Tag 0954 & Tag 0955]. □

Let λ and μ be compatible, meaning that $\nu := \mu - \lambda$ lies in the image of $X^*(\mathrm{Res}_{L/\mathbb{Q}_p}(\mathbb{T})_E) \to (E \otimes_{\mathbb{Q}_p} \mathrm{Lie}\, T)^*, \eta \mapsto d\eta$; see [JLS24, Defs. 2.4.2 & 2.4.3]. If $\bar{\nu} \in W.\nu$ is the dominant weight in the Weyl orbit of ν, let $L(\bar{\nu})$ be the corresponding algebraic representation of $\mathrm{Res}_{L/\mathbb{Q}_p}(\mathbb{G})_E$ and of G.

Lemma B.5. *Let $V \in D_\square(\mathcal{D}^{la}(G, E))$ be $\mathfrak{z}$-finite. Then $L(\bar{\nu}) \otimes_{E_\square}^L V$ is also $\mathfrak{z}$-finite.*

Proof. We can assume $V = V_{\mathfrak{m}}$ for some $\mathfrak{m}$. Then $V \in D_\square(\mathcal{D}^{la}(G, E)/\mathfrak{m}^N)$ for some $N \geq 1$, and $L(\bar{\nu}) \otimes_{E_\square}^L V = (L(\bar{\nu}) \otimes_{E_\square}^L U(\mathfrak{g})/\mathfrak{m}^N) \otimes_{U(\mathfrak{g})/\mathfrak{m}^N}^{L\square} V$. Here we view $L(\bar{\nu}) \otimes_{E_\square}^L U(\mathfrak{g})/\mathfrak{m}^N$ as a $U(\mathfrak{g})$-bimodule as in [BG80, § 2.2]. The desired statement now follows from the proof of [BG80, Cor. 2.6]. □

As in [JLS24, Def. 2.4.5], for every $\mathfrak{z}$-finite solid $\mathcal{D}^{la}(G, E)$-module V put

$$\mathrm{T}_\lambda^\mu(V) := \mathrm{pr}_{|\mu|}(L(\bar{\nu}) \otimes_{E_\square}^L \mathrm{pr}_{|\lambda|}(V)).$$

Let w_0 be the longest Weyl element and put $\lambda^\vee = -w_0\lambda$. If λ is dominant algebraic, then $\lambda^\vee$ is the highest weight of the dual representation $L(\lambda)^\vee$. Let ι be the antipode of $U(\mathfrak{g})$, namely the anti-automorphism determined by $\iota(X) = -X$ for $X \in \mathfrak{g}$. Then $\chi_{\lambda^\vee} = \chi_\lambda \circ \iota$ (see for example [Dix77, Prop. 7.4.9]).

Lemma B.6. *For the left and right $\mathcal{D}^{la}(G,E)$-actions $(g_1 \cdot f \cdot g_2)(g) = f(g_1^{-1} g g_2^{-1})$ on $\mathcal{C}_c^{la}(G,E)$, the left and right actions of $\mathfrak{z}$ agree.*

Proof. First, we have [Jan03, II.4.20, Remark]

$$\mathcal{O}(\mathrm{Res}_{L/\mathbb{Q}_p}(\mathbb{G})_E) \simeq \bigoplus_{\eta} L(\eta)^\vee \otimes_E L(\eta),$$

where η ranges over the dominant weights in $\mathrm{Res}_{L/\mathbb{Q}_p}(\mathbb{T})_E$. The left action is on $L(\eta)^\vee$, while the right action on $L(\eta)$ is $v \cdot g_2 = g_2^{-1} v$. Thus both actions of $z \in \mathfrak{z}$ on this summand are given by multiplying $\chi_{\eta^\vee}(z) = \chi_\eta(\iota(z))$.

The subspace of locally algebraic functions of $\mathcal{C}_c^{la}(G,E)$, spanned by functions of the form φf, with $\varphi \in \mathcal{C}_c^{\mathrm{sm}}(G,E)$ and $f \in \mathcal{O}(\mathrm{Res}_{L/\mathbb{Q}_p}(\mathbb{G})_E)$, is dense in $\mathcal{C}_c^{la}(G,E)$ for its locally convex topology. By continuity we get the claim. □

Proposition B.7. *For a $\mathfrak{z}$-finite $V \in \mathrm{Rep}_{E_\square}^{la}(G)$ in the $|\lambda|$-block, the objects $V^\vee = R\underline{\mathrm{Hom}}_E(V,E)$, $\mathbb{D}_{\mathrm{ST}}(V)$, and $\mathbb{D}_{\mathrm{BZ}}(V)$ lie in the $|\lambda^\vee|$-block, and there are natural equivalences in $D_\square(\mathcal{D}^{la}(G,E))$*

$$(\mathrm{T}_\lambda^\mu(V))^\vee \simeq \mathrm{T}_{\lambda^\vee}^{\mu^\vee}(V^\vee),$$
$$\mathbb{D}_{\mathrm{ST}}(\mathrm{T}_\lambda^\mu(V)) \simeq \mathrm{T}_{\lambda^\vee}^{\mu^\vee}(\mathbb{D}_{\mathrm{ST}}(V)),$$
$$\mathbb{D}_{\mathrm{BZ}}(\mathrm{T}_\lambda^\mu(V)) \simeq \mathrm{T}_{\lambda^\vee}^{\mu^\vee}(\mathbb{D}_{\mathrm{BZ}}(V)).$$

Proof. We have $(L(\bar\nu) \otimes_E V)^\vee \simeq L(\bar\nu)^\vee \otimes_E V^\vee$. Also for any weight λ, we have $(\mathrm{pr}_{|\lambda|}V)^\vee \simeq \mathrm{pr}_{|\lambda^\vee|}(V^\vee)$ under the $\mathfrak{z}$-finite assumption. Together with $L(\bar\nu)^\vee \simeq L(-w_0\bar\nu)$, this proves the first equivalence in the statement.

By definition, $\mathbb{D}_{\mathrm{ST}}(W) = (W^\vee)^{Rla}$. Applying $(-)^{Rla}$ to the first equivalence and using [RJRC23, Cor. 3.2.14(3)] gives the second equivalence.

For Bernstein–Zelevinsky duality, let $W \in \mathrm{Rep}_{E_\square}^{la}(G)$ be $\mathfrak{z}$-finite. We may assume $W \in D_\square(\mathcal{D}^{la}(G,E)/\mathfrak{m}_\lambda^N), N \geq 1$. Then we have

$$\begin{aligned} R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(W, \mathcal{C}_c^{la}(G,E)) &\simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)/\mathfrak{m}_\lambda^N}(W, R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(\mathcal{D}^{la}(G,E)/\mathfrak{m}_\lambda^N, \mathcal{C}_c^{la}(G,E))) \\ &\simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)/\mathfrak{m}_\lambda^N}(W, R\underline{\mathrm{Hom}}_{\mathfrak{z}}(\mathfrak{z}/\mathfrak{m}_\lambda^N, \mathcal{C}_c^{la}(G,E))). \end{aligned}$$

Lemma B.6 shows that $R\underline{\mathrm{Hom}}_{\mathfrak{z}}(\mathfrak{z}/\mathfrak{m}_\lambda^N, \mathcal{C}_c^{la}(G,E))$ is a $\mathfrak{z}/\mathfrak{m}_\lambda^N$-module for the right action of G. Converting this right action to a left action via the involution ι, using Lemma B.4, we see $\mathbb{D}_{\mathrm{BZ}}(W)$ is $\mathfrak{z}$-finite and

$$\mathbb{D}_{\mathrm{BZ}}(\mathrm{pr}_{|\lambda|}(W)) \simeq \mathrm{pr}_{|\lambda^\vee|}(\mathbb{D}_{\mathrm{BZ}}(W)). \tag{B.1}$$

Finally, we have

$$\begin{aligned} \mathbb{D}_{\mathrm{BZ}}(L(\bar\nu) \otimes_{E_\square}^L W) &\simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}(G,E)}(W, L(\bar\nu)^\vee \otimes_{E_\square} \mathcal{C}_c^{la}(G,E)) \\ &\simeq L(\bar\nu)^\vee \otimes_{E_\square}^L \mathbb{D}_{\mathrm{BZ}}(W). \end{aligned} \tag{B.2}$$

The second equivalence is induced by sending $f \in L(\bar\nu)^\vee \otimes_{E_\square} \mathcal{C}_c^{la}(G,E) \simeq \mathcal{C}_c^{la}(G, L(\bar\nu)^\vee)$ to $g \mapsto g^{-1} f(g)$. We get the desired equivalence by (B.1). □

Let Z_G be the center of G. If $\zeta_{\bar\nu} : Z_G \to E^\times$ is the central character of $L(\bar\nu)$, then T_λ^μ sends representations with central character ω to representations with central character $\omega\zeta_{\bar\nu}$. Let $\omega : Z_G \to E^\times$ be a locally analytic character. Put

$$\mathcal{D}_\omega^{la}(G,E) := \mathcal{D}^{la}(G,E) \otimes_{\mathcal{D}^{la}(Z_G,E),\omega}^L E.$$

Since the quotient $G \to G/Z_G$ admits locally analytic sections, one directly checks that the object $\mathcal{D}_\omega^{la}(G,E)$ is concentrated in degree 0. Write $\mathrm{Rep}_{E_\square,\omega}^{la}(G)$ for the full subcategory of $D_\square(\mathcal{D}_\omega^{la}(G,E))$ consisting of objects which are derived locally analytic. Let $\mathcal{C}_{c,\omega}^{la}(G,E)$ be the space of locally analytic functions on G with compact support modulo Z_G satisfying $f(zg) = \omega(z)^{-1} f(g)$ for $z \in Z_G$ and $g \in G$. The action $(g_1 \cdot f \cdot g_2)(g) = f(g_1^{-1} g g_2^{-1})$ makes it a $\mathcal{D}_\omega^{la}(G,E)$-bimodule.

Definition B.8. The Schneider–Teitelbaum duality with central character ω is

$$\begin{aligned} \mathbb{D}_{\mathrm{ST},\omega} : (\mathrm{Rep}_{E_\square,\omega}^{la}(G))^{\mathrm{op}} &\longrightarrow \mathrm{Rep}_{E_\square,\omega^{-1}}^{la}(G), \\ V &\longmapsto (V^\vee)^{Rla}. \end{aligned}$$

The Bernstein–Zelevinsky duality with central character ω is

$$\begin{aligned}\mathbb{D}_{\mathrm{BZ},\omega} : (\mathrm{Rep}^{la}_{E_\square,\omega}(G))^{\mathrm{op}} &\longrightarrow D_\square(\mathcal{D}^{la}_{\omega^{-1}}(G,E)),\\ V &\longmapsto R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}_{\omega}(G,E)}(V, \mathcal{C}^{la}_{c,\omega}(G,E)).\end{aligned}$$

Proposition B.9. *Let λ and μ be compatible weights, $\nu = \mu - \lambda$, and let $V \in \mathrm{Rep}^{la}_{E_\square,\omega}(G)$ be $\mathfrak{z}$-finite and in the $|\lambda|$-block. Then, in $D_\square(\mathcal{D}^{la}_{\omega^{-1}\zeta_{\bar\nu}^{-1}}(G,E))$, there are natural equivalences, for $? \in \{\mathrm{ST}, \mathrm{BZ}\}$,*

$$\mathbb{D}_{?,\omega\zeta_{\bar\nu}}(\mathrm{T}^{\mu}_{\lambda}(V)) \simeq \mathrm{T}^{\mu^\vee}_{\lambda^\vee}(\mathbb{D}_{?,\omega}(V)).$$

Proof. The equivalence for the Schneider–Teitelbaum duality follows from the proof of Proposition B.7. For Bernstein–Zelevinsky duality, similarly as above, we have

$$\begin{aligned}\mathbb{D}_{\mathrm{BZ},\omega\zeta_{\bar\nu}}(L(\bar\nu)\otimes^L_{E_\square} V) &\simeq R\underline{\mathrm{Hom}}_{\mathcal{D}^{la}_{\omega}(G,E)}(V, L(\bar\nu)^\vee \otimes_{E_\square} \mathcal{C}^{la}_{c,\omega\zeta_{\bar\nu}}(G,E))\\ &\simeq L(\bar\nu)^\vee \otimes^L_{E_\square} \mathbb{D}_{\mathrm{BZ},\omega}(V).\end{aligned} \tag{B.3}$$

Then we can conclude as in the proof of Proposition B.7. □

References

[AGN25] Piotr Achinger, Sally Gilles, and Wiesława Nizioł. Compactly supported p-adic pro-étale cohomology of analytic varieties. arXiv:2501.13651, 2025.

[AL25] Tomoyuki Abe and Christopher Lazda. Proper pushforwards on analytic adic spaces. *The Quarterly Journal of Mathematics*, 76(1):147–183, 2025.

[ALY23] Piotr Achinger, Marcin Lara, and Alex Youcis. Geometric arcs and fundamental groups of rigid spaces. *Journal für die reine und angewandte Mathematik*, 799:57–107, 2023.

[And21] Grigory Andreychev. Pseudocoherent and perfect complexes and vector bundles on analytic adic spaces. arXiv:2105.12591, 2021.

[Aok25] Ko Aoki. The sheaves–spectrum adjunction. arXiv:2302.04069, 2025.

[AS24] Shishir Agrawal and Matthias Strauch. Resolutions for locally analytic representations. arXiv:2409.05079, 2024.

[AW23] Konstantin Ardakov and Simon Wadsley. Global sections of equivariant line bundles on the p-adic upper half plane. arXiv:2312.12395, 2023. Submitted.

[Bal00] Paul Balmer. Triangular Witt groups. Part I: The 12-term localization exact sequence. *K-Theory*, 19(4):311–363, 2000.

[Ber92] Joseph Bernstein. Notes of lectures on Representations of p-adic Groups. `https://www.math.tau.ac.il/~bernstei/Unpublished_texts/unpublished_texts/Bernstein93new-harv.lect.from-chic.pdf`, 1992. Harvard University lectures, fall 1992; notes by Karl E. Rumelhart.

[BG80] Joseph N Bernstein and Sergei I Gelfand. Tensor products of finite and infinite dimensional representations of semisimple Lie algebras. *Compositio Mathematica*, 41(2):245–285, 1980.

[BGR84] Siegfried Bosch, Ulrich Güntzer, and Reinhold Remmert. *Non-Archimedean Analysis: A Systematic Approach to Rigid Analytic Geometry*, volume 261 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1984.

[BH06] Colin J. Bushnell and Guy Henniart. *The Local Langlands Conjecture for* GL(2), volume 335 of *Grundlehren der mathematischen Wissenschaften*. Springer, Berlin, 2006.

[BK90] Alexei I Bondal and Mikhail M Kapranov. Representable functors, Serre functors, and mutations. *Mathematics of the USSR-Izvestiya*, 35(3):519–541, 1990.

[BL23] Arthur Bartels and Wolfgang Lück. Algebraic K-theory of reductive p-adic groups. arXiv:2306.03452v2, 2023.

[Bod26] Andreas Bode. Six operations for $\widehat{\mathcal{D}}$-modules on rigid analytic spaces. *Selecta Mathematica. New Series*, 32, 2026. Article 31.

[Bos14] Siegfried Bosch. *Lectures on formal and rigid geometry*, volume 2105. Springer, 2014.

[Bos23a] Guido Bosco. On the p-adic pro-étale cohomology of Drinfeld symmetric spaces. arXiv:2110.10683, 2023.

[Bos23b] Guido Bosco. Rational p-adic Hodge theory for rigid-analytic varieties. arXiv:2306.06100, 2023.

[BQ26] Christophe Breuil and Zicheng Qian. *Splitting and Making Explicit the de Rham Complex of the Drinfeld Space*, volume 2397 of *Lecture Notes in Mathematics*. Springer, 2026.

[Bre15] Christophe Breuil. Vers le socle localement analytique pour GL_n II. *Mathematische Annalen*, 361:741–785, 2015.

[Bre19] Christophe Breuil. Ext^1 localement analytique et compatibilité local-global. *American Journal of Mathematics*, 141:611–703, 2019.

[BS07] Christophe Breuil and Peter Schneider. First steps towards p-adic Langlands functoriality. *Journal für die reine und angewandte Mathematik*, 610:149–180, 2007.

[CDN20] Pierre Colmez, Gabriel Dospinescu, and Wiesława Nizioł. Cohomologie p-adique de la tour de Drinfeld: le cas de la dimension 1. *Journal of the American Mathematical Society*, 33(2):311–362, 2020.

[CDN23] Pierre Colmez, Gabriel Dospinescu, and Wiesława Nizioł. Factorisation de la cohomologie étale p-adique de la tour de Drinfeld. *Forum of Mathematics, Pi*, 11:e16, 2023. 62 pp.

[CGN26] Pierre Colmez, Sally Gilles, and Wiesława Nizioł. Arithmetic duality for p-adic pro-étale cohomology of analytic curves. *Compositio Mathematica*, 162(11):2423–2495, 2026.

[Chi90] Bruno Chiarellotto. Duality in rigid analysis. In Francesco Baldassarri, Siegfried Bosch, and Bernard Dwork, editors, *p-adic analysis*, volume 1454 of *Lecture Notes in Mathematics*, pages 142–172. Springer, Berlin, 1990. Proceedings of the International Conference held in Trento, Italy, May 29–June 2, 1989.

[Col10] Pierre Colmez. Représentations de $\mathrm{GL}_2(\mathbb{Q}_p)$ et (φ,Γ)-modules. *Astérisque*, 330:281–509, 2010.

[Col16] Pierre Colmez. Représentations localement analytiques de $\mathrm{GL}_2(\mathbb{Q}_p)$ et (φ,Γ)-modules. *Representation Theory*, 20:187–248, 2016.

[Col19] Pierre Colmez. Correspondance de Langlands locale p-adique et changement de poids. *Journal of the European Mathematical Society*, 21(3):797–838, 2019.

[Con99] Brian Conrad. Irreducible components of rigid spaces. *Annales de l'Institut Fourier*, 49(2):473–541, 1999.

[CS22] Dustin Clausen and Peter Scholze. Condensed mathematics and complex geometry. arXiv:2605.11731, 2022. Revised version, 2026, of the 2022 lecture notes.

[Dat07] Jean-François Dat. Théorie de Lubin–Tate non-abélienne et représentations elliptiques. *Inventiones Mathematicae*, 169(1):75–152, 2007.

[Din22] Yiwen Ding. Locally analytic Ext^1 for $\mathrm{GL}_2(\mathbb{Q}_p)$ in de Rham non-trianguline case. *Representation Theory*, 26:122–133, 2022.

[Din23] Yiwen Ding. Change of weights for locally analytic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. arXiv:2307.04332, 2023.

[Din24] Yiwen Ding. Towards the wall-crossing of locally $\mathbb{Q}_p$-analytic representations of $\mathrm{GL}_n(K)$ for a p-adic field K. arXiv:2404.06315, 2024.

[Dix77] Jacques Dixmier. *Enveloping Algebras*. Akademie-Verlag, Berlin, 1977.

[DLB17] Gabriel Dospinescu and Arthur-César Le Bras. Revêtements du demi-plan de Drinfeld et correspondance de Langlands p-adique. *Ann. of Math. (2)*, 186(2):321–411, 2017.

[Dos12] Gabriel Dospinescu. Actions infinitésimales dans la correspondance de Langlands locale p-adique. *Mathematische Annalen*, 354(2):627–657, 2012.

[DPS23] Gabriel Dospinescu, Vytautas Paškūnas, and Benjamin Schraen. Gelfand–Kirillov dimension and the p-adic Jacquet–Langlands correspondence. *Journal für die reine und angewandte Mathematik*, 801:57–114, 2023.

[Dri74] V. G. Drinfel'd. Elliptic modules. *Mathematics of the USSR-Sbornik*, 23(4):561–592, 1974.

[Dri76] V. G. Drinfel'd. Coverings of p-adic symmetric regions. *Functional Analysis and its Applications*, 10(2):107–115, 1976.

[EGH25] Matthew Emerton, Toby Gee, and Eugen Hellmann. An introduction to the categorical p-adic Langlands program. In *The Langlands program. II. Geometrization of the Langlands correspondence*, volume 112.2 of *Proceedings of Symposia in Pure Mathematics*, pages 167–419. American Mathematical Society, 2025.

[Eme06] Matthew Emerton. Jacquet modules of locally analytic representations of p-adic reductive groups I. Construction and first properties. *Annales scientifiques de l'École Normale Supérieure*, 39(5):775–839, 2006.

[Eme17] Matthew Emerton. *Locally analytic vectors in representations of locally p-adic analytic groups*, volume 248, no. 1175 of *Memoirs of the American Mathematical Society*. American Mathematical Society, 2017.

[Far] Laurent Fargues. Dualité de Poincaré et involution de Zelevinsky dans la cohomologie équivariante des espaces rigides. `https://webusers.imj-prg.fr/~laurent.fargues/Prepublications.html`. Unfinished.

[FGL08] Laurent Fargues, Alain Genestier, and Vincent Lafforgue. *L'isomorphisme entre les tours de Lubin–Tate et de Drinfeld*, volume 262 of *Progress in Mathematics*. Birkhäuser, Basel, 2008.

[GYD18] Dennis Gaitsgory and Alexander Yom Din. An analog of the Deligne–Lusztig duality for $(\mathfrak{g}, K)$-modules. *Advances in Mathematics*, 333:212–265, 2018.

[Ham24] Linus Hamann. Zelevinsky duality on basic local Shimura varieties. *Mathematical Research Letters*, 31(2):441–458, 2024. Revised version with corrections and clarifications, arXiv:2109.01213v3, 7 October 2026.

[Han26] David Hansen. Serre functors for quasicoherent and ind-coherent sheaves. `http://www.davidrenshawhansen.net/serredualrevised.pdf`, 2026.

[HM24] Claudius Heyer and Lucas Mann. 6-functor formalisms and smooth representations. arXiv:2410.13038, 2024.

[HT01] Michael Harris and Richard Taylor. *The Geometry and Cohomology of Some Simple Shimura Varieties*, volume 151 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001.

[Hub96] Roland Huber. *Étale cohomology of rigid analytic varieties and adic spaces*, volume E30 of *Aspects of Mathematics*. Vieweg, Braunschweig, 1996.

[HW22] Yongquan Hu and Haoran Wang. On the mod p cohomology for GL_2: the non-semisimple case. *Cambridge Journal of Mathematics*, 10(2):261–431, 2022.

[HW24] Yongquan Hu and Haoran Wang. On some p-adic and mod p representations of quaternion algebra over $\mathbb{Q}_p$. *Journal für die reine und angewandte Mathematik*, 812:163–210, 2024.

[Jan03] Jens Carsten Jantzen. *Representations of algebraic groups*, volume 107 of *Math. Surv. Monogr.* Providence, RI: American Mathematical Society (AMS), 2nd ed. edition, 2003.

[JLS24] Akash Jena, Aranya Lahiri, and Matthias Strauch. Translation functors for locally analytic representations. *International Journal of Number Theory*, 20(10):2611–2640, 2024.

[Kie67a] Reinhardt Kiehl. Der Endlichkeitssatz für eigentliche Abbildungen in der nichtarchimedischen Funktionentheorie. *Inventiones mathematicae*, 2:191–214, 1967.

[Kie67b] Reinhardt Kiehl. Theorem A und Theorem B in der nichtarchimedischen Funktionentheorie. *Inventiones mathematicae*, 2:256–273, 1967.

[KS12] Jan Kohlhaase and Benjamin Schraen. Homological vanishing theorems for locally analytic representations. *Mathematische Annalen*, 353:219–258, 2012.

[Lee25] Hao Lee. Towards the derived Jacquet–Emerton module functor. *Representation Theory*, 29:659–694, 2025.

[LSW26] Zhenghui Li, Benchao Su, and Zhixiang Wu. Locally analytic vectors in the completed cohomology of quaternionic Shimura curves. arXiv:2601.13625, 2026.

[Lur17] Jacob Lurie. Higher algebra, 2017. Available at `https://www.math.ias.edu/~lurie/papers/HA.pdf`.

[Man22] Lucas Mann. A p-adic 6-functor formalism in Rigid-Analytic geometry. arXiv:2206.02022, 2022.

[Mie14] Yoichi Mieda. Zelevinsky involution and ℓ-adic cohomology of the Rapoport–Zink tower. arXiv:1401.5135, 2014.

[Mik26a] Yutaro Mikami. Finiteness and duality of cohomology of (φ, Γ)-modules and the 6-functor formalism of locally analytic representations. arXiv:2504.01780v2, 2026.

[Mik26b] Yutaro Mikami. (φ, Γ)-modules over relatively discrete algebras. arXiv:2409.14145v3, 2026.

[MM82] Yasuo Morita and Atsushi Murase. Analytic representations of SL_2 over a p-adic number field. *Journal of the Faculty of Science, University of Tokyo. Section IA*, 28(3):891–905, 1982.

[MW24] Samuel Meyer and Ferdinand Wagner. q-Hodge complexes and refined TC^-. arXiv:2410.23115v4, 2024.

[Pan26] Lue Pan. On locally analytic vectors of the completed cohomology of modular curves II. *Ann. of Math. (2)*, 203(1):121–281, 2026.

[Paš13] Vytautas Paškūnas. The image of Colmez's Montreal functor. *Publications Mathématiques de l'IHÉS*, 118:1–191, 2013.

[RC24] Juan Esteban Rodríguez Camargo. The analytic de Rham stack in rigid geometry. arXiv:2401.07738, 2024.

[RC26a] Juan Esteban Rodríguez Camargo. Geometric Sen theory over rigid analytic spaces. *Journal of the European Mathematical Society*, 28:3385–3451, 2026.

[RC26b] Juan Esteban Rodríguez Camargo. Locally analytic representations and geometric representation theory. `https://sites.google.com/view/jerodriguezcamargo/courses-and-seminars`, 2026. Lecture notes, University of Bonn, summer semester 2026.

[RC26c] Juan Esteban Rodríguez Camargo. Notes on solid geometry. arXiv:2603.03012, 2026.

[RJRC22] Joaquín Rodrigues Jacinto and Juan Esteban Rodríguez Camargo. Solid locally analytic representations of p-adic Lie groups. *Representation Theory of the American Mathematical Society*, 26(31):962–1024, 2022.

[RJRC23] Joaquín Rodrigues Jacinto and Juan Esteban Rodríguez Camargo. Solid locally analytic representations. arXiv:2305.03162v5, 2023. Version 5, revised 3 August 2026.

[RZ96] Michael Rapoport and Thomas Zink. *Period Spaces for p-divisible Groups*, volume 141 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1996.

[Sch11] Benjamin Schraen. Représentations localement analytiques de $\mathrm{GL}_3(\mathbb{Q}_p)$. *Annales scientifiques de l'École Normale Supérieure*, 44(1):43–145, 2011.

[Sch18] Peter Scholze. On the p-adic cohomology of the Lubin–Tate tower. *Annales scientifiques de l'École Normale Supérieure*, 51(4):811–863, 2018. With an appendix by Michael Rapoport.

[Sch19] Peter Scholze. Lectures on condensed mathematics. arXiv:2605.03658, 2019. Updated version, 2026; based on lectures at the University of Bonn, joint with Dustin Clausen.

[Sch25] Peter Scholze. Six-functor formalisms. arXiv:2510.26269, 2025.

[Ser55] Jean-Pierre Serre. Faisceaux algébriques cohérents. *Annals of Mathematics*, 61(2):197–278, 1955.

[Ser75] Jean-Pierre Serre. *Algèbre locale, multiplicités: cours au Collège de France, 1957–1958*, volume 11 of *Lecture Notes in Mathematics*. Springer, Berlin, 3 edition, 1975.

[SS91] Peter Schneider and Ulrich Stuhler. The cohomology of p-adic symmetric spaces. *Inventiones Mathematicae*, 105:47–122, 1991.

[SS97] Peter Schneider and Ulrich Stuhler. Representation theory and sheaves on the Bruhat–Tits building. *Publ. Math. Inst. Hautes Études Sci.*, 85:97–191, 1997.

[SS16] Tobias Schmidt and Matthias Strauch. Dimensions of some locally analytic representations. *Representation Theory*, 20:14–38, 2016.

[ST02a] Peter Schneider and Jeremy Teitelbaum. Locally analytic distributions and p-adic representation theory, with applications to GL_2. *Journal of the American Mathematical Society*, 15(2):443–468, 2002.

[ST02b] Peter Schneider and Jeremy Teitelbaum. p-adic boundary values. In Pierre Berthelot, Jean-Marc Fontaine, Luc Illusie, Kazuya Kato, and Michael Rapoport, editors, *Cohomologies p-adiques et applications arithmétiques (I)*, number 278 in Astérisque, pages 51–125. Société mathématique de France, 2002.

[ST03] Peter Schneider and Jeremy Teitelbaum. Algebras of p-adic distributions and admissible representations. *Inventiones Mathematicae*, 153(1):145–196, 2003.

[ST05] Peter Schneider and Jeremy Teitelbaum. Duality for admissible locally analytic representations. *Representation Theory*, 9(10):297–326, 2005.

[Sta25] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu`, 2025.

[Ste24] Rustam Steingart. On the higher analytic vectors of $\mathbf{B}_e$. arXiv:2410.18805, 2024.

[Su25] Benchao Su. On the locally analytic Ext^1-conjecture in the $\mathrm{GL}_2(L)$ case. arXiv:2504.17683, 2025.

[SW25] Matthias Strauch and Zhixiang Wu. Bernstein–Zelevinsky duality for locally analytic principal series representations. arXiv:2501.04850, 2025.

[vdP92] Marius van der Put. Serre duality for rigid analytic spaces. *Indagationes Mathematicae*, 3(2):219–235, 1992.

[Wei94] Charles A Weibel. *An introduction to homological algebra*. Number 38 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1994.

[Zhu25] Xinwen Zhu. Tame categorical local Langlands correspondence. arXiv:2504.07482v1, 2025.

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