GEOMETRIC TRANSLATIONS OF (φ, Γ)-MODULES FOR $\mathrm{GL}_2(\mathbb{Q}_p)$

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Abstract. We study "change of weights" maps between loci of the stack of (φ, Γ)-modules over the Robba ring with integral Hodge-Tate-Sen weights. We show that in the $\mathrm{GL}_2(\mathbb{Q}_p)$ case these maps can realize translations of (φ, Γ)-modules geometrically. The motivation is to investigate translations of locally analytic representations under the categorical p-adic Langlands correspondence.

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1. Introduction

In this introduction we consider translations of locally analytic representations of p-adic Lie groups from the point of view of the categorical p-adic Langlands program proposed by Emerton-Gee-Hellmann in [EGH23]. We will give hints for the categorical story in the $\mathrm{GL}_2(\mathbb{Q}_p)$ case by realizing Ding's result in [Din23] on translations of (φ, Γ)-modules in arithmetic families.

1.1. **Translations for locally analytic representations.** Translation is a fundamental tool to study modules over a reductive lie algebra $\mathfrak{g}$, an operation changing infinitesimal characters. Let us take $\mathfrak{g} = \mathfrak{gl}_n, n \geq 2$ to be the Lie algebra of GL_n over a p-adic coefficient field L for a prime number p with the Cartan subalgebra $\mathfrak{t}$ of the diagonal matrices. Let $U(\mathfrak{g})$ be the universal enveloping algebra and $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. Via the Harish-Chandra isomorphism an (infinitesimal) character $\chi_\lambda : Z(\mathfrak{g}) \to L$ is determined by a weight $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathfrak{t}^*$. We can consider the category $\mathrm{Mod}(U(\mathfrak{g}))_{\chi_\lambda}$ of $U(\mathfrak{g})$-modules which are generalized eigenspaces for the action of $Z(\mathfrak{g})$ of eigenvalues given by χ_λ. For another $\mu \in \mathfrak{t}^*$ such that $\lambda - \mu \in \mathbb{Z}^n$ is integral, the translation operator gives a functor

$$T_\lambda^\mu : \mathrm{Mod}(U(\mathfrak{g}))_{\chi_\lambda} \to \mathrm{Mod}(U(\mathfrak{g}))_{\chi_\mu}.$$

If λ and μ are both dominant integral and have the same regularity (in the sense of the stabilizers in the Weyl group for the dot action), T_λ^μ induces an equivalence of categories. While translations between regular and non-regular characters (into and out of the walls) are more interesting.

Locally analytic representations of a p-adic Lie group G, say $G = \mathrm{GL}_n(\mathbb{Q}_p)$, are naturally $\mathfrak{g}$-modules by differentiating the G-actions. Under p-adic Langlands correspondence, infinitesimal characters of locally analytic representations correspond to generalized Hodge-Tate(-Sen) weights of the associated p-adic Galois representations, cf. [DPS20]. Translations for locally analytic representations were studied by Jena-Lahiri-Strauch in [JLS21]. If a locally analytic representation π is in $\mathrm{Mod}(U(\mathfrak{g}))_{\chi_\lambda}$, then its translation $T_\lambda^\mu \pi$ is still a locally analytic representation, with generalized infinitesimal character χ_μ. It is then extremely interesting to investigate how translations intertwine with the Langlands correspondence. The operations that change weights were already

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observed by Colmez in [Col18]. A more systematic study was carried out by Ding in [Din23], based on Colmez's construction $D \mapsto D^\natural \boxtimes \mathbf{P}^1$ of p-adic local Langlands [Col10, Col16] from rank two (φ, Γ)-modules over the Robba ring to locally analytic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. Ding's idea is to translate (φ, Γ)-modules firstly which can be equipped with $\mathfrak{g}$-actions using Colmez's method (by the infinitesimal action of G on $D = D \boxtimes \mathbb{Z}_p \subset D \boxtimes \mathbf{P}^1$). Ding proposed recently in [Din24] conjectures to study p-adic Langlands correspondences for general GL_n via translation functors.

1.2. **Categorical p-adic Langlands conjecture.** Let Rig_L be the category of rigid analytic spaces over L. Emerton-Gee-Hellmann consider in [EGH23] the moduli stack $\mathfrak{X}_n$ (over Rig_L) of (φ, Γ)-modules of rank n over the Robba ring, which should be viewed as the p-adic analytic version of the stack of Langlands parameters for $\mathrm{GL}_n(\mathbb{Q}_p)$. Let $\mathrm{D}^b_{\mathrm{f.p}}(\mathrm{an}.G)$ be the derived category of locally analytic representations of $G = \mathrm{GL}_n(\mathbb{Q}_p)$ (with conjectural finiteness condition discussed in [EGH23, §6.2]) and let $D^b_{\mathrm{Coh}}(\mathfrak{X}_n)$ be the derived category of coherent sheaves on $\mathfrak{X}_n$. The analytic version of the categorical p-adic Langlands correspondence predicts the existence of a functor

$$\mathfrak{A}^{\mathrm{rig}}_G : \mathrm{D}^b_{\mathrm{f.p}}(\mathrm{an}.G) \to D^b_{\mathrm{Coh}}(\mathfrak{X}_n)$$

which should satisfy various properties, particularly including the compatibility between infinitesimal characters and Hodge-Tate-Sen weights.

Let $\mathbf{h} = (h_1, \cdots, h_n) \in \mathbb{Z}^n, h_1 \leq \cdots \leq h_n$ be fixed integral Sen weights and $\lambda = \lambda_{\mathbf{h}} := (h_n - (n-1), \cdots, h_i - (i-1), \cdots, h_1)$ be the corresponding (automorphic) weight of $\mathfrak{t}$. Let $\mathrm{D}^b_{\mathrm{f.p}}(\mathrm{an}.G)_{\chi_\lambda} \subset \mathrm{D}^b_{\mathrm{f.p}}(\mathrm{an}.G)$ be the full subcategory consisting of representations with generalized infinitesimal character χ_λ. We consider the substack $(\mathfrak{X}_n)^\wedge_{\mathbf{h}}$ (appeared in [EGH23, §5.3.22]), the formal completion of $\mathfrak{X}_n$ along the weight $\mathbf{h}$ locus. For an affinoid algebra A, the A-value of $(\mathfrak{X}_n)^\wedge_{\mathbf{h}}$ is the groupoid of (φ, Γ)-modules D_A of rank n over $\mathrm{Sp}(A)$ such that for any point $x \in \mathrm{Sp}(A)$, the specialization $D_A \otimes_A k(x)$ has Sen weights $\mathbf{h}$. Then $\mathfrak{A}^{\mathrm{rig}}_G$ should restrict to a functor:

$$\mathfrak{A}^{\mathrm{rig}}_G : \mathrm{D}^b_{\mathrm{f.p}}(\mathrm{an}.G)_{\chi_\lambda} \to D^b_{\mathrm{Coh}}((\mathfrak{X}_n)^\wedge_{\mathbf{h}}).$$

For different integral weights $\lambda_{\mathbf{h}}, \lambda_{\mathbf{h}'}$, the composite of $\mathfrak{A}^{\mathrm{rig}}_G$ and the translation functor $T^{\lambda_{\mathbf{h}'}}_{\lambda_{\mathbf{h}}}$: $\mathrm{D}^b_{\mathrm{f.p}}(\mathrm{an}.G)_{\chi_{\lambda_{\mathbf{h}}}} \to \mathrm{D}^b_{\mathrm{f.p}}(\mathrm{an}.G)_{\chi_{\lambda_{\mathbf{h}'}}}$ translates sheaves on $(\mathfrak{X}_n)^\wedge_{\mathbf{h}}$ to $(\mathfrak{X}_n)^\wedge_{\mathbf{h}'}$. If one believes in an ultimate equivalence of categories statement of the categorical p-adic Langlands correspondence as Fargues-Scholze (see [EGH23, Rem. 1.4.6]), it is then natural to ask if there exists a morphism between spaces $(\mathfrak{X}_n)^\wedge_{\mathbf{h}}$ and $(\mathfrak{X}_n)^\wedge_{\mathbf{h}'}$ (that induces translations of sheaves).

1.3. **Change of weights.** The functor $\mathfrak{A}^{\mathrm{rig}}_G$ is in conjectural and the geometric properties of $\mathfrak{X}_n$ are largely unknown. However, the answer to the question above is positive. We suppose that $\mathbf{h}$ is regular for simplicity[1] and let $0 = (0, \cdots, 0)$ be the zero weight. Let B be the Borel subgroup of upper triangular matrices of GL_n with Lie algebra $\mathfrak{b}$. Consider the Grothendieck resolution

$$f : \widetilde{\mathfrak{g}} = \mathrm{GL}_n \times^B \mathfrak{b} = \{(\nu, gB) \in \mathfrak{g} \times \mathrm{GL}_n/B \mid \mathrm{Ad}(g^{-1})(\nu) \in \mathfrak{b}\} \to \mathfrak{g}, (\nu, gB) \mapsto \nu$$

where Ad denotes the adjoint action.

Proposition 1.1 (Proposition 3.12)**.** *There exists a (change of weights) morphism of stacks*

$$f_{\mathbf{h}} : (\mathfrak{X}_n)^\wedge_{\mathbf{h}} \to (\mathfrak{X}_n)^\wedge_0 \tag{1.1}$$

such that the following commutative diagram of stacks over Rig_L

$$\begin{array}{ccc} (\mathfrak{X}_n)^\wedge_{\mathbf{h}} & \xrightarrow{D_{\mathrm{pdR}}} & \widetilde{\mathfrak{g}}/\mathrm{GL}_n \\ \downarrow f_{\mathbf{h}} & & \downarrow f \\ (\mathfrak{X}_n)^\wedge_0 & \xrightarrow{D_{\mathrm{pdR}}} & \mathfrak{g}/\mathrm{GL}_n \end{array} \tag{1.2}$$

is Cartesian.

The morphisms D_{pdR} are the local model maps defined in *loc. cit.* It firstly sends a (φ, Γ)-module D to the associated B^+_{dR}-representation $W^+_{\mathrm{dR}}(D)$ of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ which, using Fontaine's classification, gives a rank n bundle $D_{\mathrm{pdR}}(D)$ with a (Hodge) filtration $\mathrm{Fil}^\bullet D_{\mathrm{pdR}}(D)$ stabilized under a nilpotent linear endomorphism ν. The filtration depends on the regularity of Sen weights and is parametrized by the stack $*/P = (\mathrm{GL}_n/P)/\mathrm{GL}_n$ where GL_n/P is a flag variety and $P = B$,

[1]Namely $h_1 < \cdots < h_n$. Proposition 1.1 works for general $\mathrm{GL}_n(K)$ and non-regular $\mathbf{h}$ with suitable modifications.

resp. $P=\mathrm{GL}_n$, in the case of weight $\mathbf{h}$, resp. 0. In the language of B-pairs of Berger [Ber08a], the map $f_{\mathbf{h}}$, already pointwisely described in [Din24, Lem. 2.1], sends the B-pair (W_e, W_{dR}^+) attached to D to $(W_e, W_{\mathrm{dR},0}^+)$ where $W_{\mathrm{dR},0}^+$ is the unique $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-invariant B_{dR}^+-lattice inside $W_{\mathrm{dR}}(D)=W_{\mathrm{dR}}^+(D)[\frac{1}{t}]$ of weight 0 associated to the trivial filtration of $D_{\mathrm{pdR}}(D)$. The proof of the isomorphism

$$(\mathfrak{X}_n)_{\mathbf{h}}^{\wedge}\simeq(\mathfrak{X}_n)_0^{\wedge}\times_{\mathfrak{g}/\mathrm{GL}_n}\widetilde{\mathfrak{g}}/\mathrm{GL}_n$$

is a simple combination of the known family versions of the equivalence between (φ,Γ)-modules and B-pairs and the classification of B_{dR}^+-representations with integral weights (see Appendix A).

Remark 1.2. The condition for a (φ,Γ)-module D with integral weights being de Rham is equivalent to the vanishing of the nilpotent endomorphism ν on $D_{\mathrm{pdR}}(D)=D_{\mathrm{dR}}(D)$. Let $\mathfrak{X}_n^{\mathrm{DE}}$ be the stack of rank n de Rham (φ,Γ)-modules of weight zero (so called p-adic differential equations). The restriction of the diagram (1.2) to $\nu=0$ locus is

$$\begin{array}{ccc}\mathfrak{X}_n^{\mathrm{DE}}\times_{*/\mathrm{GL}_n}*/B & \longrightarrow & */B=(\mathrm{GL}_n/B)/\mathrm{GL}_n\\ \Big\downarrow{\scriptstyle f_{\mathbf{h}}} & & \Big\downarrow{\scriptstyle f}\\ \mathfrak{X}_n^{\mathrm{DE}} & \longrightarrow & */\mathrm{GL}_n.\end{array}$$

Thus $\mathfrak{X}_n^{\mathrm{DE}}\times_{*/\mathrm{GL}_n}*/B$ is isomorphic to the stack of de Rham (φ,Γ)-modules of weight $\mathbf{h}$. On the other hand, using Berger's equivalence [Ber08b], this stack is locally isomorphic to $\mathrm{WD}_n\times_{*/\mathrm{GL}_n}*/B$ ([EGH23, Thm. 5.2.4]) where WD_n is the analytification of the stack of Weil-Deligne representations of rank n.

An immediate consequence of Proposition 1.1 is the existence of isomorphisms between loci of $\mathfrak{X}_n$ with different regular Hodge-Tate-Sen weights. We will use $f_{\mathbf{h}}$ to realize translations of (φ,Γ)-modules in $\mathrm{GL}_2(\mathbb{Q}_p)$-case and then discuss a general speculation.

1.4. **Geometric translations of (φ,Γ)-modules.** Now we focus on the case $G=\mathrm{GL}_2(\mathbb{Q}_p)$ where we have Colmez's construction. The main result of this paper can only be stated and proved in this case. Following Colmez, there is a unique way to make a (φ,Γ)-module D_A over an affinoid $\mathrm{Sp}(A)$ a $\mathfrak{g}$-module so that $Z(\mathfrak{g})$ acts via a character determined by the Sen weights of D_A, cf. [Dos12]. If $D_A\in(\mathfrak{X}_2)_{\mathbf{h}}^{\wedge}(A)$ for some fixed weight $\mathbf{h}$ with associate $\lambda=\lambda_{\mathbf{h}}\in\mathfrak{t}^*$, then one can talk about the translation $T_\lambda^\mu D_A$ to another integral weight as $\mathfrak{g}$-modules. Ding's method shows that $T_\lambda^\mu D_A$ is still a (φ,Γ)-module. The following is our main theorem.

Theorem 1.3 (Theorem 5.15). *Suppose* $\mathbf{h}=(h_1,h_2)\in\mathbb{Z}^2, h_1<h_2,\ \lambda=\lambda_{\mathbf{h}},\ \mu=\lambda_0=(-1,0)$. *Let* $D_{(\mathfrak{X}_2)_{\mathbf{h}}^{\wedge}}$ *(resp.* $D_{(\mathfrak{X}_2)_0^{\wedge}}$*) be the restriction of the universal* (φ,Γ)*-module on* $\mathfrak{X}_2$ *to* $(\mathfrak{X}_2)_{\mathbf{h}}^{\wedge}$ *(resp. to* $(\mathfrak{X}_2)_0^{\wedge}$*). Then the following statements are true locally on affinoid charts of* $\mathfrak{X}_2$ *(namely on any* $\mathrm{Sp}(A)$ *with a formally smooth map* $\mathrm{Sp}(A)\to\mathfrak{X}_2$*).*

(1) There exists an isomorphism

$$T_\lambda^\mu D_{(\mathfrak{X}_2)_{\mathbf{h}}^{\wedge}}\simeq f_{\mathbf{h}}^* D_{(\mathfrak{X}_2)_0^{\wedge}}$$

of (φ,Γ)*-modules of rank two which induces the map* $f_{\mathbf{h}}:(\mathfrak{X}_2)_{\mathbf{h}}^{\wedge}\to(\mathfrak{X}_2)_0^{\wedge}$.

(2) There exists an isomorphism

$$T_\mu^\lambda D_{(\mathfrak{X}_2)_0^{\wedge}}\simeq Rf_{\mathbf{h},*}D_{(\mathfrak{X}_2)_{\mathbf{h}}^{\wedge}}$$

of (φ,Γ)*-modules of rank four and in degree* 0 *on* $(\mathfrak{X}_2)_0^{\wedge}$.

Certainly, all objects in the above theorem need proper definitions. The theorem will be stated and proved without the language of stacks. For a chart $\mathrm{Sp}(A)\to\mathfrak{X}_2$ with D_A the pullback of the universal (φ,Γ)-module, we construct the space $\mathrm{Sp}(A)^\wedge=\mathrm{Sp}(A)\times_{\mathfrak{X}_2}(\mathfrak{X}_2)_0^{\wedge}$ as what should be called an affinoid formal rigid space. Using Proposition 1.1, we only need to prove the results for the map $f_{\mathbf{h}}^{-1}(\mathrm{Sp}(A)^\wedge)=\mathrm{Sp}(A)^\wedge\times_{\mathfrak{g}/\mathrm{GL}_2}\widetilde{\mathfrak{g}}/\mathrm{GL}_2\to\mathrm{Sp}(A)^\wedge$ between formal rigid spaces and (φ,Γ)-modules over these spaces. Fortunately, the map is proper and the cohomologies can be studied via GAGA theorems (see Appendix B).

The key of the proof is the geometric properties of the Grothendieck resolution $f:\widetilde{\mathfrak{g}}\to\mathfrak{g}$. For example, $Rf_*\mathcal{O}_{\widetilde{\mathfrak{g}}}$ concentrates in degree 0 and is locally free of rank two over $\mathcal{O}_{\mathfrak{g}}$, which is basically the reason that $Rf_{\mathbf{h},*}D_{(\mathfrak{X}_2)_{\mathbf{h}}^{\wedge}}$ has rank four and concentrates in degree 0. Another vital input is the flatness of the local model map D_{pdR} (to use flat base change). We can prove the flatness of D_{pdR} in $\mathrm{GL}_2(\mathbb{Q}_p)$ case (§3.4) and we expect it is always flat for other G.

1.5. **Speculation.** Finally, we explain the motivation of Theorem 1.3. For $U(\mathfrak{g})$-modules with a generalized infinitesimal character, translation functors can be realized geometrically using Beilinson-Bernstein localization, cf. [BG99]. A better approach for translations (into and out of the wall) is to consider singular localizations which send $\mathrm{Mod}(U(\mathfrak{g}))_{\chi_\mu}$ for non-regular χ_μ to some D-modules on corresponding partial flag varieties G/P [BK15, BMR06]. Then translation functors after localization can be realized using pushforward and pullback along the maps between flag varieties like $G/B \to G/P$, see [BK15, §6] and more similarly [BMR06, Lem. 2.2.5].

The functor $\mathfrak{A}_G^{\mathrm{rig}}$ in §1.2 is expected to be certain localization, of the form ([EGH23, Rem. 6.2.9])

$$\pi \mapsto \mathcal{L}_\infty \widehat{\otimes}^L_{\mathcal{D}(G)} \pi$$

where $\mathcal{D}(G)$ is the distribution algebra of G and $\mathcal{L}_\infty$ plays the role of the sheaf of differential operators on $\mathfrak{X}_n$. As in Proposition 1.1, we fix a regular Hodge-Tate weight $\mathbf{h}$ (resp. non-regular weight 0) and let χ_λ (resp. χ_μ) be the associated infinitesimal character. With the map (1.1), we get a diagram of functors

$$\begin{array}{ccc} \mathrm{D}^b_{\mathrm{f.p}}(\mathrm{an}.G)_{\chi_\lambda} & \xrightarrow{\mathfrak{A}_G^{\mathrm{rig}}} & D^b_{\mathrm{Coh}}((\mathfrak{X}_n)^\wedge_{\mathbf{h}}) \\ T_\mu^\lambda \uparrow \downarrow T_\lambda^\mu & & Lf_{\mathbf{h}}^* \uparrow \downarrow Rf_{\mathbf{h},*} \\ \mathrm{D}^b_{\mathrm{f.p}}(\mathrm{an}.G)_{\chi_\mu} & \xrightarrow{\mathfrak{A}_G^{\mathrm{rig}}} & D^b_{\mathrm{Coh}}((\mathfrak{X}_n)^\wedge_0). \end{array}$$

Question 1.4. *In the above diagram, do we have* $\mathfrak{A}_G^{\mathrm{rig}} \circ T_\lambda^\mu = Rf_{\mathbf{h},*} \circ \mathfrak{A}_G^{\mathrm{rig}}$ *and* $\mathfrak{A}_G^{\mathrm{rig}} \circ T_\mu^\lambda = Lf_{\mathbf{h}}^* \circ \mathfrak{A}_G^{\mathrm{rig}}$*?*

Taking account of the adjunction for translation functors, this suggests to ask whether we have isomorphisms (where w_0 denotes the longest element in the Weyl group)

$$T_{-w_0\lambda}^{-w_0\mu} \mathcal{L}_\infty|_{(\mathfrak{X}_n)^\wedge_{\mathbf{h}}} \simeq Lf_{\mathbf{h}}^* \mathcal{L}_\infty|_{(\mathfrak{X}_n)^\wedge_0}, \tag{1.3}$$

$$T_{-w_0\mu}^{-w_0\lambda} \mathcal{L}_\infty|_{(\mathfrak{X}_n)^\wedge_0} \simeq Rf_{\mathbf{h},*} \mathcal{L}_\infty|_{(\mathfrak{X}_n)^\wedge_{\mathbf{h}}}? \tag{1.4}$$

Remark 1.5. The sheaf $\mathcal{L}_\infty$ should be a family version of the dual of $\Pi(D)$ where for a (φ,Γ)-module D of rank n, we write $\Pi(D)$ for the conjectural locally analytic representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ attached to D via p-adic local Langlands correspondence. The expected isomorphism $T_{-w_0\lambda}^{-w_0\mu} \mathcal{L}_\infty|_{(\mathfrak{X}_n)^\wedge_{\mathbf{h}}} \simeq Lf_{\mathbf{h}}^* \mathcal{L}_\infty|_{(\mathfrak{X}_n)^\wedge_0}$ is just a family (dual) version of a conjecture of Ding [Din24, Conj. 1.1, (1)]: if D has Hodge-Tate-Sen weights $\mathbf{h}$, then $T_\lambda^\mu \Pi(D) = \Pi(f_{\mathbf{h}}(D))$.

At present, there is no construction of the sheaf $\mathcal{L}_\infty$ for $n \geq 2$. In the case of $\mathrm{GL}_2(\mathbb{Q}_p)$, as in the Banach case [EGH23, §7.3], $\mathcal{L}_\infty$ should be the family version of Colmez's construction $D \mapsto D^\natural \boxtimes \mathbf{P}^1$ from (φ,Γ)-modules to $\mathcal{D}(G)$-modules (up to a twist). An easier object to construct is $D \boxtimes \mathbf{P}^1$ as only $U(\mathfrak{g})$-modules, which equals to copies of (φ,Γ)-modules. The main theorem immediately implies the following (compared with (1.3) and (1.4) up to a twist).

Corollary 1.6 (Corollary 5.24)**.** *In the notation of Theorem 1.3, the following isomorphisms of sheaves of* $U(\mathfrak{g})$*-modules hold locally on affinoid charts of* $\mathfrak{X}_2$ *with suitable definitions of the objects:*

$$T_\lambda^\mu D_{(\mathfrak{X}_2)^\wedge_{\mathbf{h}}} \boxtimes \mathbf{P}^1 \simeq Lf_{\mathbf{h}}^* D_{(\mathfrak{X}_2)^\wedge_0} \boxtimes \mathbf{P}^1,$$
$$T_\mu^\lambda D_{(\mathfrak{X}_2)^\wedge_0} \boxtimes \mathbf{P}^1 \simeq Rf_{\mathbf{h},*} D_{(\mathfrak{X}_2)^\wedge_{\mathbf{h}}} \boxtimes \mathbf{P}^1.$$

Remark 1.7. In the trianguline cases, we expect that the identification of functors in Question 1.4 applying for finite slope Orlik-Strauch representations is compatible with the conjectural description of $\mathfrak{A}_G^{\mathrm{rig}}(\pi)$ using local models and Bezrukavnikov's functor in [EGH23, §6.2.25], provided the version of *loc. cit.* for non-regular weights (see [Wu21] for a discussion on cycles).

Remark 1.8. Geometric translations for real local Langlands correspondence were already discussed to some extent, cf. [ABV92, §16], [Str14], etc.

1.6. **Outline.** We review basics on Grothendieck-Springer resolutions in §2. In §3, we study the change of weights maps and prove Proposition 1.1, and show that the local model map is flat in the $\mathrm{GL}_2(\mathbb{Q}_p)$-case. In §4, we compute translations of (φ,Γ)-modules in families. We prove the main theorem on geometric translations in §5. We also show in §5.3 how to recover some of Ding's pointwise calculation of translations from our main Theorem 1.3. In two appendices A and B, we collect facts on families of (φ,Γ)-modules over the Robba rings and formal rigid geometry.

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1.8. **Notation.** We fix a prime number p. We use L a finite extension of $\mathbb{Q}_p$ as the coefficient field.

Suppose that K is a p-adic local field. Let $K_\infty = K(\mu_\infty) = \cup_m K(\mu_{p^m})$ be the extension of K by adding all p-th power roots of unity, and $\Gamma_K := \mathrm{Gal}(K_\infty/K)$. Also let $K_m = K(\mu_{p^m})$ for $m \geq 1$. Let $\mathcal{G}_K = \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ be the absolute Galois group of K. We write $\Gamma = \Gamma_K$ if $K = \mathbb{Q}_p$.

Let ϵ be the cyclotomic character of $\mathcal{G}_K$ viewed also as the character $\mathrm{Norm}_{K/\mathbb{Q}_p}|\mathrm{Norm}_{K/\mathbb{Q}_p}|_p$ of $K^\times$ where $|p|_p = p^{-1}$. In our convention, the cyclotomic character has Hodge-Tate weights one.

We follow the notation of (φ,Γ)-modules in [KPX14]. If X is a rigid space, we write $\mathcal{R}_{X,K}$ for the Robba ring of K over X [KPX14, Def. 6.2.1]. If A is an affinoid algebra, write $\mathcal{R}_{A,K} := \mathcal{R}_{\mathrm{Sp}(A),K}$. We also need the Robba rings $\mathcal{R}^r_{A,K}$ and $\mathcal{R}^{[s,r]}_{A,K}$ for $0 < s < r$ small enough in the sense of [KPX14, Def. 2.2.2]. The ring $\mathcal{R}^{[s,r]}_{\mathbb{Q}_p,K}$ is the ring of single variable rigid analytic functions converging over $\mathbb{U}^{[s,r]}_{K_0'} = \{p^{-\frac{r}{p-1}} \leq |X| \leq p^{-\frac{s}{p-1}}\}$ with coefficients in K_0' the maximal unramified subfield of K_∞ and $\mathcal{R}^r_{\mathbb{Q}_p,K} = \varprojlim_{s\leq r} \mathcal{R}^{[s,r]}_{\mathbb{Q}_p,K}$ is the functions on $\mathbb{U}^r_{K_0'} = \cup_{0<s<r}\mathbb{U}^{[s,r]}_{K_0'}$. Then $\mathcal{R}^{[s,r]}_{A,K} = \mathcal{R}^{[s,r]}_{\mathbb{Q}_p,K}\widehat{\otimes}_{\mathbb{Q}_p}A$ and $\mathcal{R}^r_{A,K} = \varprojlim_{s\leq r}\mathcal{R}^{[s,r]}_{A,K}$. The group Γ_K acts on $\mathcal{R}^r_{A,K}$ and $\varphi : \mathcal{R}^r_{A,K} \to \mathcal{R}^{r/p}_{A,K}$.

A (φ,Γ_K)-module D_A over $\mathcal{R}_{A,K}$ is always the base change of a (φ,Γ_K)-module D^r_A over $\mathcal{R}^r_{A,K}$ for some r small enough: a finite projective $\mathcal{R}^r_{A,K}$-module D^r_A equipped with an isomorphism $\varphi^* D^r_A = \mathcal{R}^{r/p}_{A,K} \otimes_{\varphi,\mathcal{R}^r_{A,K}} D^r_A \simeq \mathcal{R}^{r/p}_{A,K} \otimes_{\mathcal{R}^r_{A,K}} D^r_A$ and a commuting continuous semilinear action of Γ_K. Given a (φ,Γ_K)-module D^r_A over $\mathcal{R}^r_{A,K}$, we write $D^s_A := \mathcal{R}^s_{A,K} \otimes_{\mathcal{R}^r_{A,K}} D^r_A, D^{[s,r]}_A := \mathcal{R}^{[s,r]}_{A,K} \otimes_{\mathcal{R}^r_{A,K}} D^r_A, D_A := \mathcal{R}_{A,K} \otimes_{\mathcal{R}^r_{A,K}} D^r_A$ for $s \leq r$. The (φ,Γ_K)-cohomologies $H^i_{\varphi,\gamma_K}(D_A), i = 0,1,2$ for a (φ,Γ_K)-module is defined using Herr complex as in [KPX14, Def. 2.3.3] where γ_K is a fixed topological generator of Γ_K modulo the torsion subgroup. And we write $\mathrm{Hom}_{\varphi,\gamma_K}(-,-)$ for the Hom space of (φ,Γ_K)-modules. We write $\mathcal{R}_A = \mathcal{R}_{A,K}$, $\gamma = \gamma_K$, etc., if $K = \mathbb{Q}_p$.

Let $B^+_{\mathrm{dR}}, B_{\mathrm{dR}} = B^+_{\mathrm{dR}}[\frac{1}{t}]$ be Fontaine's de Rham period rings, where t is Fontaine's $2\pi i$.

For any $r > 0$, let $m(r)$ be the minimal integer such that $p^{m(r)-1}[K(\mu_{p^\infty}) : K_0(\mu_{p^\infty})]r \geq 1$ where K_0 is the maximal unramified subfield of K. The integer is taken so that there are injections $\iota_m : \mathcal{R}^r_{L,K} \hookrightarrow (L \otimes_{\mathbb{Q}_p} K_m)[[t]]$ for $m \geq m(r)$, see Appendix A.

For a continuous character $\delta : K^\times \to \Gamma(X,\mathcal{O}_X)^\times$ over a rigid space X, write $\mathcal{R}_{X,K}(\delta)$ for the corresponding rank one (φ,Γ_K)-module over $\mathcal{R}_{X,K}$ in [KPX14, Cons. 6.2.4]. If $K = \mathbb{Q}_p$, write $z : \mathbb{Q}_p^\times \hookrightarrow L^\times$ for the algebraic character of weight one. Recall that $\mathcal{R}_L(z) = t\mathcal{R}_L$.

If $G = \mathrm{GL}_n$ over L for some n, we always take the Borel B the subgroup of upper-triangular matrices and T the diagonal torus. We use the fraktur letter $\mathfrak{g}$ (resp. $\mathfrak{b}$, resp. $\mathfrak{t}$) for the Lie algebra of the group G (resp. B, resp. T), also viewed as an affine scheme (or its analytification) over L. Denote by $\mathrm{Ad} : G \to \mathrm{End}(\mathfrak{g})$ the adjoint representation. For a Lie algebra $\mathfrak{g}$, denote by $U(\mathfrak{g})$ the universal enveloping algebra and $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$. We write $\mathcal{N} \subset \mathfrak{g}$ for the nilpotent cone.

If X is a rigid space with $x \in X$, we write $k(x)$ for the residue field at x. For a (φ,Γ_K)-module D_X over X, we write D_x or $D_{k(x)}$ for the base change to x. More generally for an affinoid algebra A and an A-point $\mathrm{Sp}(A) \to X$, write $D_A = D_X \otimes_{\mathcal{R}_{X,K}} \mathcal{R}_{A,K}$, and similarly $D^r_A = D^r_X \otimes_{\mathcal{R}^r_{X,K}} \mathcal{R}^r_{A,K}$, etc.

Let $\mathcal{C}_L$ denote the category of commutative local Artinian L-algebras with residue field L. If $A \in \mathcal{C}_L$, let $\mathfrak{m}_A$ be its maximal ideal.

If Z is a commutative ring, I is a finitely generated ideal of Z and M is a Z-module, then write $M[I] = \{m \in M \mid z.m = 0, \forall z \in I\}$ and $M[I^\infty] := \cup_{i=1}^\infty M[I^i]$. If $\chi : Z \to A$ is a surjction of rings with kernel I and M is an A-module, write $M[Z = \chi] = M[I]$ and $M\{Z = \chi\} = M[I^\infty]$.

2. The Grothendieck-Springer resolution

In this section, we recall the basics of the Grothendieck-Springer resolution. We only consider schemes over L for the moment.

Let G be a split reductive group over L with a Borel subgroup B and a maximal torus T. We will write $\mathfrak{h} = \mathfrak{t}$ for the Cartan subalgebra. We use P to denote standard parabolic subgroups containing B with the Lie algebra $\mathfrak{p}$. Write W (resp. W_P) for the Weyl group of G (resp. the Levi of P).

2.1. **Recollection on Grothendieck-Springer resolution.** For a parabolic subgroup $P \supset B$ with the Levi subgroup M containing T, consider the scheme

$$\widetilde{\mathfrak{g}}_P := G \times^P \mathfrak{p} \simeq \{(\nu, gP) \in \mathfrak{g} \times G/P \mid \mathrm{Ad}(g^{-1})\nu \in \mathfrak{p}\}$$

and the partial Grothendieck resolution

$$f_P : \widetilde{\mathfrak{g}}_P \to \mathfrak{g}, (\nu, gP) \mapsto \nu.$$

We will omit the subscript P when $P = B$, namely write $f : \widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_B \to \mathfrak{g}$. There is a natural map

$$\widetilde{\mathfrak{g}}_P \to \mathfrak{m}//M \simeq \mathfrak{h}/W_P$$

sending (ν, gP) to the projection of $\mathrm{Ad}(g^{-1})\nu \in \mathfrak{p}$ to $\mathfrak{m}$. The map is compatible with f_P in the sense that it induces a map $g_P : \widetilde{\mathfrak{g}}_P \to \mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h}/W_P$.

Lemma 2.1. *Let f_P, g_P be as above.*

(1) The morphism f_P is proper and surjective, finite over $\mathfrak{g}^{\mathrm{reg}}$ and is finite étale of degree $|W/W_P|$ over $\mathfrak{g}^{\mathrm{reg-ss}}$. Here $\mathfrak{g}^{\mathrm{reg-ss}} \subset \mathfrak{g}^{\mathrm{reg}} \subset \mathfrak{g}$ denote open subschemes of regular semisimple elements and regular elements.

(2) The natural map $\mathcal{O}_{\mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h}/W_P} \to Rg_{P,*}\mathcal{O}_{\widetilde{\mathfrak{g}}_P}$ is an isomorphism. Moreover, the map $\mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h}/W_P \to \mathfrak{g}$ is finite flat of rank $|W/W_P|$.

Proof. (1) See for example [Wu21, Lem. 2.3].

(2) The isomorphism is [BK15, Lem. 3.2]. The map $\mathfrak{h}/W_P \to \mathfrak{h}/W$ is flat by miracle flatness. □

We don't really need the following lemma, but it might be helpful to keep it in mind.

Lemma 2.2. *The dualizing sheaf of $\widetilde{\mathfrak{g}}_P$ is trivial. And there is a canonical isomorphism $f_P^!\mathcal{F} = Lf_P^*\mathcal{F}$ for any coherent $\mathcal{O}_{\mathfrak{g}}$-module $\mathcal{F}$.*

Proof. We follow the proof of [BK07, Lemma. 5.1.1]. Let $\pi_P : \widetilde{\mathfrak{g}}_P \to G/P$ be the projection. The canonical bundle $\omega_{\widetilde{\mathfrak{g}}_P}$ of $\widetilde{\mathfrak{g}}_P$ is isomorphic to $\pi_P^*\omega_{G/P} \otimes_{\mathcal{O}_{\widetilde{\mathfrak{g}}_P}} \omega_{\pi_P}$ where ω_{π_P} denotes the relative dualizing sheaf. We know $\omega_{G/P} = G \times^P \delta_P$ where δ_P is the sum of all roots in the unipotent radical $\mathfrak{u}$ of $\mathfrak{p}$. Moreover, since $\widetilde{\mathfrak{g}}_P = G \times^P \mathfrak{p}$, $\omega_{\pi_P} = G \times^P \omega_{\mathfrak{p}}$. Consider the fiberation $0 \to \mathfrak{u} \to \mathfrak{p} \to \mathfrak{m} \to 0$. Choosing coordinates m, u of $\mathfrak{m}$ and $\mathfrak{u}$, then ω_P is generated by $dm \wedge du$. Since $\omega_{\mathfrak{m}}$ is trivial as a M-representation, $\omega_{\mathfrak{p}} = -\delta_P$ as a P-representation. Hence $\omega_{\widetilde{\mathfrak{g}}_P} \simeq \mathcal{O}_{\widetilde{\mathfrak{g}}_P}$.

We get $f_P^!\mathcal{O}_{\mathfrak{g}} = f_P^!\omega_{\mathfrak{g}} = \mathcal{O}_{\widetilde{\mathfrak{g}}_P}$. The map $f_P : \widetilde{\mathfrak{g}}_P \to \mathfrak{g}$ is perfect, hence $f_P^! \simeq Lf_P^*$ by [Sta24, Tag 0B6U,Tag 068D]. □

2.2. **Direct images of some line bundles.** We need compute direct images of some line bundles on $\widetilde{\mathfrak{g}}$ in the $G = \mathrm{GL}_2$ case (Proposition 2.4). The computation will be the key for our main result on direct images of (φ, Γ)-modules (Proposition 5.10).

Take $\mathfrak{g} = \mathfrak{gl}_2 = \mathrm{Spec}(L[a, b, c, z])$ where a, b, c, z are coordinates for entries of matrices

$$\begin{pmatrix} a+z & b \\ c & -a+z \end{pmatrix} \in \mathfrak{g}.$$

Take $\mathfrak{h} = \mathrm{Spec}(L[h, z])$ for $(z+h, z-h) \in \mathfrak{h}$. Recall the map $g : \widetilde{\mathfrak{g}} \to \mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h}$ where $\mathfrak{h}/W = L[z, h^2]$ and $\mathcal{O}(\mathfrak{h}/W) \to \mathcal{O}(\mathfrak{g}) : h^2 \mapsto a^2 + bc$.

Lemma 2.3. *The map $f : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is a finite étale rank two cover over $\mathfrak{g}^{\mathrm{reg-ss}} = \mathfrak{g} \setminus \{a^2 + bc = 0\}$ and is finite flat of degree two over $\mathfrak{g}^{\mathrm{reg}} = \mathfrak{g} \setminus \{a = b = c = 0\}$. The fiber of f over $0 \in \mathfrak{g}$ is identified with G/B and the fiber over a closed point $x \in \mathcal{N} \setminus \{0\}$ is ramified of degree 2 over the residue field $k(x)$.*

Proof. By [BHS19, Prop. 2.1.1] or Lemma 2.1, we know the corresponding preimage $\widetilde{\mathfrak{g}}^{\mathrm{reg-ss}}$ (resp. $\widetilde{\mathfrak{g}}^{\mathrm{reg}}$) is finite étale (resp. finite) over $\mathfrak{g}^{\mathrm{reg-ss}}$ (resp. $\mathfrak{g}^{\mathrm{reg}}$). Since $f_*\mathcal{O}_{\widetilde{\mathfrak{g}}^{\mathrm{reg}}} = \mathcal{O}_{\mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h} \setminus \{a=b=c=0\}}$, we see $\widetilde{\mathfrak{g}}^{\mathrm{reg}} = (\mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h}) \setminus \{a = b = c = 0\}$. The fiber over $x \in \mathcal{N} \setminus \{0\} \subset \mathfrak{g}^{\mathrm{reg}}$, which has image 0 in $\mathfrak{h}/W$, is the fiber of $\mathfrak{h} \to \mathfrak{h}/W$ over 0 whose coordinate ring is $k(x)[h]/h^2$. □

Let $\pi : \widetilde{\mathfrak{g}} \to G/B$. For $k \in \mathbb{Z}$, we write $\mathcal{O}_{\widetilde{\mathfrak{g}}}(k)$ for the line bundle $\pi^*\mathcal{O}_{G/B}(k)$ where our convention is the standard one that $\mathcal{O}_{G/B}(k)$ is ample for $k \geq 1$. Let $\mathcal{V} = \mathcal{O}_{\mathfrak{g}}e_1 \oplus \mathcal{O}_{\mathfrak{g}}e_2$ be the universal trivialized rank two bundle on $\mathfrak{g}$. The operator ν acts universally on $\mathcal{V}$

$$\nu : x_1e_1 + x_2e_2 \mapsto ((a+z)x_1 + bx_2)e_1 + (cx_1 + (z-a)x_2)e_2$$

for $x_1, x_2 \in \mathcal{O}_{\mathfrak{g}}$, hence also on $f^*\mathcal{V}$. By definition, ν stabilizes the following short exact sequence which gives the universal filtration on $\widetilde{\mathfrak{g}}$ pulled back from G/B (arising from the B-filtration of the standard representation of GL_2):

$$0 \to \mathcal{O}_{\widetilde{\mathfrak{g}}}(-1) \to f^*\mathcal{V} \to \mathcal{O}_{\widetilde{\mathfrak{g}}}(1) \to 0. \tag{2.1}$$

Proposition 2.4. *We write $\mathcal{U} = \mathcal{O}(\mathfrak{g}) = L[a,b,c,z]$ and $\widetilde{\mathcal{U}} := \mathcal{O}(\mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h}) = \mathcal{U}[h]/(h^2 - (a^2+cb))$ which is free of rank two over $\mathcal{U}$.*

(1) The sheaves $Rf_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(-1)$ and $Rf_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(1)$ concentrate in degree zero and are free of rank two over $\mathcal{U}$.

(2) Let $\widetilde{\mathcal{V}} := \mathcal{V} \otimes_{\mathcal{U}} \widetilde{\mathcal{U}} = f_*f^*\mathcal{V}$. The sequence of $\widetilde{\mathcal{U}}$-modules below

$$\cdots \xrightarrow{\nu-(z-h)} \widetilde{\mathcal{V}} \xrightarrow{\nu-(h+z)} \widetilde{\mathcal{V}} \xrightarrow{\nu-(z-h)} \widetilde{\mathcal{V}} \xrightarrow{\nu-(h+z)} \cdots$$

is exact. Moreover, the $\widetilde{\mathcal{U}}$-module $\widetilde{\mathcal{V}}[\nu - (z \pm h)] := \ker(\widetilde{\mathcal{V}} \xrightarrow{\nu-(z\pm h)} \widetilde{\mathcal{V}})$ is free of rank two over $\mathcal{U}$ and $(\nu - (z \pm h))\mathcal{V} = (\nu - (z \pm h))\widetilde{\mathcal{V}} = \widetilde{\mathcal{V}}[\nu - (z \mp h)]$.

(3) The sequence of $\widetilde{\mathcal{U}}$-modules

$$0 \to f_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(-1) \to \widetilde{\mathcal{V}} \to f_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(1) \to 0$$

is exact and identifies $f_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(-1)$ with $\widetilde{\mathcal{V}}[\nu - (z+h)]$.

Proof. (1) The vanishing of higher direct images follows from the same proof for $\mathcal{O}_{\widetilde{\mathfrak{g}}}$ in [BMRR08, Prop. 3.4.1] which can be deduced from vanishing results for the Steinberg resolution case, cf. [Bro93] or [BK07, Thm. 5.2.1]. In detail, by the projection formula, $R\pi_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(i) = (R\pi_*\mathcal{O}_{\widetilde{\mathfrak{g}}}) \otimes_{\mathcal{O}_{G/B}} \mathcal{O}_{G/B}(i)$. Consider the sequence of the bundles $0 \to \widetilde{\mathcal{N}} = G \times^B \mathfrak{n} \to \widetilde{\mathfrak{g}} \to G \times^B \mathfrak{h} \to 0$ on G/B where $\mathfrak{n}$ is the nilpotent cone of $\mathfrak{b}$. This short exact sequence induces a filtration on $\pi_*\mathcal{O}_{\widetilde{\mathfrak{g}}}$ with the associated graded algebra isomorphic to $\mathcal{O}(\mathfrak{h}) \otimes \mathcal{O}_{\widetilde{\mathcal{N}}}$ (see [Har13, Ex. II.5.16]). Since $\mathcal{O}_{\widetilde{\mathcal{N}}}(\pm 1)$ has vanishing higher cohomology, so is $\pi_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(\pm 1)$. More directly, we can take an increasing filtration by grading on $\mathcal{O}_{\widetilde{\mathfrak{g}}}$ with graded pieces the coherent sheaves $G \times^B \mathrm{Sym}^n \mathfrak{b}^*$ which are finite extensions of $\mathcal{O}(2i), i \geq 0$ and have vanishing higher cohomology even after twisting $\mathcal{O}_{G/B}(-1)$. Now we show that $Rf_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(\pm 1)$ are free over $\mathfrak{g}$. The dualizing complexes of $\widetilde{\mathfrak{g}}$ and $\mathfrak{g}$ are trivial by Lemma 2.2 and f is proper. Let $\mathbb{D}_{\mathrm{GS}}(-)$ be the Grothendieck-Serre duality. We have $\mathcal{O}_{\widetilde{\mathfrak{g}}}(1) = \mathbb{D}_{\mathrm{GS}}(\mathcal{O}_{\widetilde{\mathfrak{g}}}(-1))[-\dim \mathfrak{g}]$ and hence $Rf_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(1) = \mathbb{D}_{\mathrm{GS}}(Rf_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(-1))[-\dim \mathfrak{g}]$. Thus $Rf_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(\pm 1)$ are maximal Cohen-Macaulay sheaves, hence locally free, on $\mathfrak{g}$ (cf. [Sta24, Tag 0DWZ,Tag 090U]). The (generic) rank is two by Lemma 2.3.

(2) The composite $(\nu-(h-z))\circ(\nu-(h+z)) = 0$ is due to $h^2 = a^2+bc$ in $\widetilde{\mathcal{U}}$. A section $x_1e_1+x_2e_2 \in \widetilde{\mathcal{V}}, x_1, x_2 \in \widetilde{\mathcal{U}}$ is in the kernel of $\nu-(h+z)$ if and only if $(a-h)x_1+bx_2 = cx_1-(a+h)x_2 = 0$. Using that $\widetilde{\mathcal{U}}$ is free over $\mathcal{U}$ with a basis $1, h$, we write $x_i = y_i + hz_i, z_i, y_i \in \mathcal{U}, i = 1, 2$. The last condition is equivalent to $y_1 = az_1 + bz_2, y_2 = cz_1 - az_2, ay_1 - (a^2+bc)z_1 + by_2 = cy_1 - ay_2 - (a^2+bc)z_2 = 0$ and to $y_1 = az_1 + bz_2, y_2 = cz_1 - az_2$. We see that there is a $\mathcal{U}$-surjection $\mathcal{V} = \mathcal{U}^2 \twoheadrightarrow \widetilde{\mathcal{V}}[\nu - (h+z)] : (z_1, z_2) \mapsto ((a+h)z_1 + bz_2)e_1 + (cz_1 - (a-h)z_2)e_2$ which is an isomorphism (for example $(a+h)z_1 + bz_2 = 0$ implies that $z_1 = 0$). Hence the $\widetilde{\mathcal{U}}$-map $\nu - (z-h) : \widetilde{\mathcal{V}} \to \widetilde{\mathcal{V}}[\nu - (z+h)]$ is surjective. The embedding $\mathcal{V} \hookrightarrow \widetilde{\mathcal{V}}$ induces a surjection $\mathcal{V} \to \widetilde{\mathcal{V}}/(\nu - (h+z)) = \widetilde{\mathcal{V}}/(h - (\nu - z))$. Then the map $\nu - (z-h) : \widetilde{\mathcal{V}}/(\nu - (z+h)) \to \widetilde{\mathcal{V}}[\nu - (z+h)]$ is an isomorphism.

(3) We identify $\mathcal{O}_{\widetilde{\mathfrak{g}}}(-1) \subset \mathcal{V}$ with the subsheaf $\mathcal{O}_{\widetilde{\mathfrak{g}}}(g.e_1)$ where $g \in \mathrm{GL}_2(\mathcal{O}(\mathrm{GL}_2))$ denotes the universal element. The map $\widetilde{\mathfrak{g}} \to \mathfrak{h}$ sends (ν, gB) to the image of $\mathrm{Ad}(g^{-1})\nu$ in $\mathfrak{h} = \{(h+z, h-z)\}$ via $\mathfrak{b} \to \mathfrak{h}$. Hence $h + z \in \mathcal{O}(\mathfrak{h})$, pulled back to $\widetilde{\mathfrak{g}}$, satisfies that $\nu(g.e_1) = g.(\mathrm{Ad}(g^{-1})\nu)e_1 = (h+z)(g.e_1)$. Thus $\mathcal{O}_{\widetilde{\mathfrak{g}}}(-1) \subset f^*\mathcal{V}[\nu - (h+z)]$. Taking direct images we get $f_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(-1) \subset (f_*f^*\mathcal{V})[\nu - (h+z)]$. We claim that $\mathcal{O}_{\widetilde{\mathfrak{g}}}(-1) = f^*\mathcal{V}[\nu - (h+z)]$ (when restricted to $\widetilde{\mathfrak{g}}^{\mathrm{reg}}$). Then $f_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(-1) = \widetilde{\mathcal{V}}[\nu - (h+z)]$ when restricted to $\mathfrak{g}^{\mathrm{reg}}$ (Lemma 2.3), hence on whole $\mathfrak{g}$ by [Sta24, Tag 0EBJ] and that $\widetilde{\mathcal{V}}[\nu - (h+z)]$ is a vector bundle by (2). By G-equivariance, we may check the identification on the open subspace $U := \widetilde{\mathfrak{g}} \cap (\{\nu = \begin{pmatrix} a+z & b \\ c & -a+z \end{pmatrix}\} \times \{\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} B\})$. Since

$\mathrm{Ad}(g^{-1})\nu = \begin{pmatrix} a+bx+z & b \\ c-2ax-bx^2 & -a-bx+z \end{pmatrix}$, we see $h = a+bx$ and $c-2ax-bx^2=0$. A section $x_1e_1+x_2e_2 \in f^*\mathcal{V}(U), x_1, x_2 \in \mathcal{O}_{\widetilde{\mathfrak{g}}}(U)$ is in the kernel of $\nu-(h+z)$ if and only if $(a-h)x_1+bx_2 = cx_1-(a+h)x_2 = 0$ if and only if $b(x_2-xx_1) = (2a+bx)(x_2-xx_1) = 0$ in $\mathcal{O}_{\widetilde{\mathfrak{g}}}(U)$. The scheme U is integral which implies that $x_2-xx_1=0$. Then $x_1e_1+x_2e_2 = x_1(e_1+xe_2) = x_1(g.e_1) \in \mathcal{O}_{\widetilde{\mathfrak{g}}}(-1)(U)$ and we finished the proof. □

Remark 2.5. Those rank two free sheaves in Proposition 2.4 on $\mathfrak{g}$ are not flat over $\mathcal{O}_{\mathfrak{g}\times_{\mathfrak{h}/W}\mathfrak{h}}$. The fibers of these sheaves at $a=b=c=z=0$ have dimension 2 but are not free of rank one over $L[h]/h^2$. For example, by the proof of (2) of the proposition, we know

$$f_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(-1) = \widetilde{\mathcal{V}}[\nu-(h+z)] \simeq (\widetilde{\mathcal{U}}x_1 \oplus \widetilde{\mathcal{U}}x_2)/((a-h)x_1+bx_2, cx_1-(a+h)x_2).$$

3. Change of weights

We will construct (in §3.2) the change of weights maps and prove the product formula for the completions of the stack $\mathfrak{X}_n$ of (φ,Γ)-modules of rank $n \geq 1$ along fixed Sen weights loci (Proposition 3.12). In §3.3, we describe a general construction for families of (φ,Γ)-modules changing possibly non-fixed weights. Then we will study the flatness of the local model map in the $\mathrm{GL}_2(\mathbb{Q}_p)$ case in §3.4.

We fix K a finite extension of $\mathbb{Q}_p$ and assume $\Sigma = \mathrm{Hom}(K,L)$ has size $|K:\mathbb{Q}_p|$. Take $\mathbf{h} = (\mathbf{h}_\sigma)_{\sigma\in\Sigma} = (h_{\sigma,1},\cdots,h_{\sigma,n})_{\sigma\in\Sigma} \in (\mathbb{Z}^n)^\Sigma$ such that $h_{\sigma,1} \leq \cdots \leq h_{\sigma,n}$ for all σ. Let $G = \prod_{\sigma\in\Sigma}\mathrm{GL}_{n/L}$ with the Weyl group $W \simeq (\mathcal{S}_n)^\Sigma$ where $\mathcal{S}_n$ is the n-th symmetric group and with the Lie algebra $\mathfrak{g}$. Let $P_{\mathbf{h}} = \prod_{\sigma\in\Sigma} P_{\mathbf{h}_\sigma}$ be the standard parabolic subgroup of G containing $\prod_\sigma B$ such that the Weyl group $W_{P_{\mathbf{h}}}$ of the Levi subgroup of $P_{\mathbf{h}}$ is the stabilizer subgroup of $\mathbf{h}$ for the action of $W = \mathcal{S}_n^\Sigma$ on $(\mathbb{Z}^n)^\Sigma$. We write $\widetilde{\mathfrak{g}}_{\mathbf{h}}$ for $\widetilde{\mathfrak{g}}_{P_{\mathbf{h}}}$ and $f_{\mathbf{h}} = f_{P_{\mathbf{h}}} : \widetilde{\mathfrak{g}}_{P_{\mathbf{h}}} \to \mathfrak{g}$, the analytification of the map in §2.

3.1. Stacks of almost de Rham (φ,Γ)-modules.

We recall the setting in [EGH23, §5] and the definition of various stacks.

Let Rig_L be the category of rigid analytic spaces over L equipped with the Tate-fpqc topology defined in [CT09, §2.1]. By a stack we mean a category fibered in groupoids over Rig_L satisfying descent for the Tate-fpqc topology. Given a stack $\mathfrak{X}$ over Rig_L and $\mathrm{Sp}(A) \in \mathrm{Rig}_L$, we write $\mathfrak{X}(A) := \mathfrak{X}(\mathrm{Sp}(A)) = \mathrm{Hom}(\mathrm{Sp}(A),\mathfrak{X})$ for the groupoid lying over $\mathrm{Sp}(A)$ (cf. Yoneda lemma [Sta24, Tag 0GWI,Tag 02XY]). Sheaves with values in sets are viewed as stacks fibered in discrete categories.

Example 3.1. If $Y \in \mathrm{Rig}_L$, then Y defines a sheaf over Rig_L via Yoneda embedding by [Con06, Cor. 4.2.5]. Let $Z \subset Y \in \mathrm{Rig}_L$ be a Zariski-closed subspace and let $\mathcal{I}$ be the coherent ideal sheaf. We define a subsheaf $Y^\wedge \subset Y$ such that for any $X \in \mathrm{Rig}_L$, $Y^\wedge(X)$ is the set of morphisms $X \to Y$ such that the image of X lies set-theoretically in Z. For $n \in \mathbf{N}$, let $\mathfrak{Y}_n$ be the closed subspace of Y cut out by $\mathcal{I}^n$. We get a directed system $\cdots\mathfrak{Y}_n \hookrightarrow \mathfrak{Y}_{n+1}\cdots$ of sheaves over Rig_L. Then $Y^\wedge$ is equal to the sheaf colimit $\varinjlim_n \mathfrak{Y}_n$: for an affinoid $\mathrm{Sp}(A) \in \mathrm{Rig}_L$, we have $Y^\wedge(A) = \varinjlim_n \mathfrak{Y}_n(A)$ (cf. [Sta24, Tag 0738,Tag 0GXT,Tag 0AIX]).

For a (φ,Γ_K)-module $D_{L'}$ over $\mathcal{R}_{L',K}$ for a finite extension L' of L, the roots of the Sen polynomial of $D_{L'}$ in $(K\otimes_{\mathbb{Q}_p} L')[T] = \prod_{\sigma\in\Sigma} L'[T]$ are Sen weights of $D_{L'}$, see [KPX14, Def. 6.2.11]. The σ-components of the roots for $\sigma\in\Sigma$ are called σ-Sen weights.

Definition 3.2.
1. Let L' be a finite extension of L, then a (φ,Γ_K)-module $D_{L'}$ of rank n over $\mathcal{R}_{L',K}$ is said to be almost de Rham (resp. has Sen weights $\mathbf{h}$) if all the Sen weights are integers (resp. for all $\sigma\in\Sigma$, the multiset of σ-Sen weights of $D_{L'}$ is equal to $\{h_{\sigma,1},\cdots,h_{\sigma,n}\}$.)
2. Let $X \in \mathrm{Rig}_L$. A (φ,Γ_K)-module D_X of rank n over $\mathcal{R}_{X,K}$ is said to be almost de Rham (resp. almost de Rham of weight $\mathbf{h}$) if for every point $x \in X$, the specialization D_x is almost de Rham (resp. has Sen weights $\mathbf{h}$).

If $X = \mathrm{Sp}(A)$, then a (φ,Γ_K)-module D_A over $\mathcal{R}_{A,K}$ is almost de Rham of weight $\mathbf{h}$ if and only if its Sen polynomial $P_{\mathrm{Sen}} \in (K\otimes_{\mathbb{Q}_p} A)[T]$ is equal to $\prod_{\sigma\in\Sigma}\prod_{i=1}^n (T-h_{\sigma,i})$ modulo the nilradical of A.

Definition 3.3. Let $(\mathfrak{X}_n)^\wedge_{\mathbf{h}}$ be the category fibered in groupoids over Rig_L sending $X \in \mathrm{Rig}_L$ to the groupoid of almost de Rham (φ,Γ_K)-modules over $\mathcal{R}_{X,K}$ of weight $\mathbf{h}$.

Definition 3.4. Assume that $X \in \mathrm{Rig}_L$. Suppose that $D_{\mathrm{pdR},X}$ is a finite projective rank n module over $\mathcal{O}_X \otimes_{\mathbb{Q}_p} K$ equipped with a decreasing filtration $\mathrm{Fil}_X^{\bullet} = (\mathrm{Fil}_X^i)_{i\in\mathbb{Z}}$ by projective $\mathcal{O}_X$-submodules of $D_{\mathrm{pdR},X}$.

(1) The filtration $\mathrm{Fil}_X^{\bullet}$ is said to be of type $\mathbf{h}$ if for any $\sigma \in \Sigma$, the filtration $\mathrm{Fil}_{X,\sigma}^{\bullet} := \mathrm{Fil}_X^{\bullet} \otimes_{\mathcal{O}_X\otimes_{\mathbb{Q}_p} K, 1\otimes\sigma} \mathcal{O}_X$ of $D_{\mathrm{pdR},X,\sigma} := D_{\mathrm{pdR},X} \otimes_{\mathcal{O}_X\otimes_{\mathbb{Q}_p} K,1\otimes\sigma} \mathcal{O}_X$ satisfies that for all $i \in \mathbb{Z}$, $\mathrm{Fil}_{X,\sigma}^{-i}/\mathrm{Fil}_{X,\sigma}^{-i+1}$ is projective over $\mathcal{O}_X$ of rank the multiplicity of i in $\{h_{\sigma,1},\cdots,h_{\sigma,n}\}$.
(2) Let $\widetilde{\mathfrak{g}}_{\mathbf{h}}/G$ be the category fibered in groupoids over Rig_L sending $X \in \mathrm{Rig}_L$ to the groupoid of triples $(D_{\mathrm{pdR,X}}, \nu_X, \mathrm{Fil}_X^{\bullet})$ where $D_{\mathrm{pdR,X}}$ is a projective $\mathcal{O}_X \otimes_{\mathbb{Q}_p} K$-module of rank n, ν_X is an $\mathcal{O}_X \otimes_{\mathbb{Q}_p} K$-linear endomorphism of $D_{\mathrm{pdR,X}}$ and $\mathrm{Fil}_X^{\bullet}$ is a filtration of $D_{\mathrm{pdR},X}$ by projective sub-$\mathcal{O}_X$-modules of type $\mathbf{h}$ stabilized by ν.
(3) Let $\widetilde{\mathfrak{g}}_{\mathbf{h}}$ be the category fibered in groupoids over Rig_L sending $X \in \mathrm{Rig}_L$ to the groupoid of $(D_{\mathrm{pdR,X}}, \nu_X, \mathrm{Fil}_X^{\bullet}, \alpha_X)$ where $(D_{\mathrm{pdR,X}}, \nu_X, \mathrm{Fil}_X^{\bullet}) \in (\widetilde{\mathfrak{g}}_{\mathbf{h}}/G)(X)$ and $\alpha_X : D_{\mathrm{pdR,X}} \simeq (\mathcal{O}_X \otimes_{\mathbb{Q}_p} K)^n$ is an isomorphism of $\mathcal{O}_X \otimes_{\mathbb{Q}_p} K$-modules.

Lemma 3.5. *The categories fibered in groupoids $(\mathfrak{X}_n)_{\mathbf{h}}^{\wedge}$ and $\widetilde{\mathfrak{g}}_{\mathbf{h}}/G$ in Definition 3.3 and 3.4 define stacks on* Rig_L.

Proof. We need to verify that $(\mathfrak{X}_n)_{\mathbf{h}}^{\wedge}$ and $\widetilde{\mathfrak{g}}_{\mathbf{h}}/G$ satisfy descent for Tate-fpqc coverings. The descents for (φ,Γ_K)-modules and $(D_{\mathrm{pdR},X}, \nu_X, \mathrm{Fil}_X^{\bullet})$ are effective by descents of vector bundles (cf. [Con06, Thm. 4.2.8]). The properties of being of weight/type $\mathbf{h}$, etc., can be checked pointwisely and thus descend. □

If $\mathbf{h} = 0$, the information on filtrations is trivial and in this case, we write $\mathfrak{g}/G$ for $\widetilde{\mathfrak{g}}_{\mathbf{h}}/G$. Let $f_{\mathbf{h}} : \widetilde{\mathfrak{g}}_{\mathbf{h}}/G \to \mathfrak{g}/G$ be the natural morphism of stacks forgetting the filtrations.

Lemma 3.6. *The stack $\widetilde{\mathfrak{g}}_{\mathbf{h}}$ is represented by the rigid analytic space $\widetilde{\mathfrak{g}}_{\mathbf{h}}$. The following diagram of stacks*

$$\begin{array}{ccc} \widetilde{\mathfrak{g}}_{\mathbf{h}} & \longrightarrow & \widetilde{\mathfrak{g}}_{\mathbf{h}}/G \\ \big\downarrow f_{\mathbf{h}} & & \big\downarrow f_{\mathbf{h}} \\ \mathfrak{g} & \longrightarrow & \mathfrak{g}/G \end{array}$$

is Cartesian.

Proof. A filtration $\mathrm{Fil}_{X,\sigma}^{\bullet}$ of type $\mathbf{h}$ of $D_{\mathrm{pdR},X,\sigma} \simeq \mathcal{O}_X$ on $X \in \mathrm{Rig}_L$ is determined by the flags $\mathrm{Fil}^{-h_{\sigma,n}} \supseteq \cdots \supseteq \mathrm{Fil}^{-h_{\sigma,1}}$ for $\sigma \in \Sigma$. With isomorphisms $D_{\mathrm{pdR},X,\sigma} \simeq \mathcal{O}_X^n$, one sees by definition that such flags are parametrized by the flag varieties $\mathrm{GL}_n/P_{\mathbf{h}_\sigma}$. An endomorphism $\nu_{X,\sigma}$ of $\mathcal{O}_X^n$ is equivalent to a map $X \to \mathfrak{gl}_n$ and the map $X \to \mathfrak{gl}_n \times \mathrm{GL}_n/P_{\mathbf{h}_\sigma}$ factors through $\widetilde{\mathfrak{gl}}_{n P_{\mathbf{h}_\sigma}}$ if and only if $\nu_{X,\sigma}$ stabilizes the filtration. The diagram is Cartesian by definitions. □

3.2. **A product formula.** We recall the local model maps and will define the change of weights maps in Proposition 3.12. We will freely use constructions and results on families of almost de Rham (φ,Γ_K)-modules in Appendix A.

Suppose $\mathrm{Sp}(A) \in \mathrm{Rig}_L$ and $D_A \in (\mathfrak{X}_n)_{\mathbf{h}}^{\wedge}(A)$. Apply Proposition A.6 for the Γ_K-representations localized from D_A in §A.1, we obtain functorially a triple

$$(D_{\mathrm{pdR}}(D_A), \nu_A, \mathrm{Fil}^{\bullet} D_{\mathrm{pdR}}(D_A)) \in (\widetilde{\mathfrak{g}}_{\mathbf{h}}/G)(A).$$

Such construction glues along admissible Tate coverings and is functorial for the base change, hence we get the following statement, which already appeared in [EGH23, §5.3.22].

Proposition 3.7. *The functor $D_{\mathrm{pdR}} : D_X \mapsto (D_{\mathrm{pdR}}(D_X), \nu_X, \mathrm{Fil}^{\bullet} D_{\mathrm{pdR}}(D_X))$ induces a (local model) morphism $D_{\mathrm{pdR}} : (\mathfrak{X}_n)_{\mathbf{h}}^{\wedge} \to \widetilde{\mathfrak{g}}_{\mathbf{h}}/G$ of stacks over* Rig_L.

Remark 3.8. The map D_{pdR} factors through the substack $(\widetilde{\mathfrak{g}}_{\mathbf{h}}/G)_0^{\wedge}$ where all ν_X are required to be locally nilpotent in the sense of Lemma 3.9 below.

Lemma 3.9. *We consider nilpotent operators.*

(1) Let A be a commutative Noetherian ring with the nilradical I. Let $\nu \in M_n(A)$ be an n-by-n matrix. Then ν is nilpotent if and only if its image in $M_n(A/I)$ satisfies that $\nu^n = 0$. And in this case, there is an integer N depending on A, n such that $\nu^N = 0$ in $M_n(A)$.

(2) *Let $X \in \mathrm{Rig}_L$ and let $\nu \in \mathrm{End}_{\mathcal{O}_X}(\mathcal{O}_X^n)$. Then ν is locally nilpotent (ν is nilpotent when restricted to any affinoid open $\mathrm{Sp}(A) \subset X$) if and only if for any $x \in X$, the image of ν in $\mathrm{End}_{k(x)}(k(x)^n) = M_n(k(x))$ is nilpotent.*

Proof. (1) Suppose that $\nu^n \in M_n(I)$. Since I is nilpotent, we see that there exists an integer N such that $I^N = 0$. Then $\nu^{nN} = (\nu^n)^N = 0$. Conversely, suppose that ν is nilpotent. We may assume that A is reduced. For any prime ideal $\mathfrak{p}$ of A, the image of ν in $M_n(A/\mathfrak{p}) \subset M_n(\mathrm{Frac}(A/\mathfrak{p}))$ is nilpotent. Hence $\nu^n \equiv 0 \mod \mathfrak{p}$ for all prime ideal $\mathfrak{p}$ of A. This implies that $\nu^n = 0$.

(2) Suppose that ν is pointwisely nilpotent. We prove that ν is locally nilpotent. The problem is local and we may assume $X = \mathrm{Sp}(A)$ is an affinoid. Let I be the nilradical of A. Then the image of ν^n in $M_n(A/I)$ is zero since it is true pointwisely. By (1), ν is nilpotent. □

The following lemma allows change of weights and is just the family version of [Din24, Lem. 2.1]. See §3.3 for a more direct construction.

Lemma 3.10. *Let $\mathrm{Sp}(A) \in \mathrm{Rig}_L$ and $D_A \in (\mathfrak{X}_n)^{\wedge}_{\mathbf{h}}(A)$. There exists a unique (φ, Γ_K)-module $f_{\mathbf{h}}(D_A) \in (\mathfrak{X}_n)^{\wedge}_0(A)$ almost de Rham of weight 0 such that $f_{\mathbf{h}}(D_A)$ is a sub-(φ, Γ_K)-module of $D_A[\frac{1}{t}]$ and $f_{\mathbf{h}}(D_A)[\frac{1}{t}] = D_A[\frac{1}{t}]$. Moreover, the formation $D_A \mapsto f_{\mathbf{h}}(D_A)$ is functorial and commutes with base change.*

Proof. By definition, there exists $r > 0$ such that $D_A = \mathcal{R}_{A,K} \otimes_{\mathcal{R}^r_{A,K}} D^r_A$ for a (φ, Γ_K)-module D^r_A over $\mathcal{R}^r_{A,K}$. We assume that $m(r)$ is large enough in the sense of Definition A.4. We construct $f_{\mathbf{h}}(D^r_A)$ firstly and will let $f_{\mathbf{h}}(D_A) = \mathcal{R}_{A,K} \otimes_{\mathcal{R}^r_{A,K}} f_{\mathbf{h}}(D^r_A)$. By Proposition A.3, the (φ, Γ_K)-modules inside $D^r_A[\frac{1}{t}]$ which equal $D^r_A[\frac{1}{t}]$ after inverting t is in bijection with Γ_K-invariant $(A \otimes_{\mathbb{Q}_p} K_m)[[t]]$-lattices in $D^m_{\mathrm{dif}}(D^r_A)$ for $m = m(r)$. By (1) of Proposition A.6 and also the Step 1 of its proof, there exists a Γ_K-invariant lattice inside $D^m_{\mathrm{dif}}(D^r_A)$ of weight 0 corresponding to the filtrations $\mathrm{Fil}^\bullet$ of type 0 of $D_{\mathrm{pdR}}(D_A)$, which can only be the trivial filtration:

$$\mathrm{Fil}^0 = D_{\mathrm{pdR}}(D_A) \supset \mathrm{Fil}^1 = \{0\}.$$

Thus there exists a sub (φ, Γ_K)-module $f_{\mathbf{h}}(D^r_A)$ of rank n inside $D^r_A[\frac{1}{t}]$ of weight 0 such that $f_{\mathbf{h}}(D^r_A)[\frac{1}{t}] = D^r_A[\frac{1}{t}]$. To show the uniqueness, suppose that $\Delta_A, \Delta'_A \subset D_A[\frac{1}{r}]$ are two required (φ, Γ_K)-submodules of weight 0. Then we obtain a map in

$$\mathrm{Hom}_{\varphi,\gamma_K}(\Delta_A, \Delta'_A[\frac{1}{t}]) = \varinjlim_{i\geq 0} \mathrm{Hom}_{\varphi,\gamma_K}(\Delta_A, t^{-i}\Delta'_A) = \mathrm{Hom}_{\varphi,\gamma_K}(\Delta_A, \Delta'_A).$$

The last equality follows from that $H^0_{\varphi,\gamma_K}(t^{-i}\Delta^\vee_A \otimes_{\mathcal{R}_{A,K}} \Delta'_A/t^{-i+1}(\Delta^\vee_A \otimes_{\mathcal{R}_{A,K}} \Delta'_A)) = 0$ for all $i \geq 1$ by Lemma 3.11 below since $\Delta^\vee_A \otimes \Delta'_A$ has weight 0. Hence the identity $\Delta_A[\frac{1}{t}] = \Delta'_A[\frac{1}{t}]$ is induced by a map $\Delta_A \to \Delta'_A$ which is necessarily an inclusion. We conclude that $\Delta_A = \Delta'_A$. The functoriality follows from the functorialities of the constructions in Appendix A or the uniqueness of $f_{\mathbf{h}}(D_A)$. □

Lemma 3.11. *Suppose that D_A is a (φ, Γ_K)-module over $\mathcal{R}_{A,K}$ for an affinoid algebra A over L and is almost de Rham of weight 0. Then $H^0_{\varphi,\gamma_K}(t^iD_A/t^{i+1}D_A) = 0$ for all $i \neq 0$.*

Proof. We know t^iD_A has weights all equal to i. Suppose that $D_A = \mathcal{R}_{A,K} \otimes_{\mathcal{R}_{A,K}} D^r_A$ for some (φ, Γ_K)-module over $\mathcal{R}^r_{A,K}$ and $r > 0$. Then $t^iD_A/t^{i+1}D_A = \varinjlim_{r'\geq r} t^iD^{r'}_A/t^{i+1}D^{r'}_A = \varinjlim_{r'\geq r} \prod_{m\geq m(r')} D^m_{\mathrm{Sen}}(t^iD_A)$, cf. Appendix A and [Liu15, Prop. 2.15]. Taking φ-invariants we have $(t^iD_A/t^{i+1}D_A)^{\varphi=1} = D^\infty_{\mathrm{Sen}}(t^iD_A) = K_\infty \otimes_{K_m} D^m_{\mathrm{Sen}}(t^iD_A)$ (cf. [KPX14, Prop. 3.2.4, Def. 6.2.11]). By definition $H^0_{\varphi,\gamma_K}(t^iD_A/t^{i+1}D_A) \subset D^\infty_{\mathrm{Sen}}(t^iD_A)^{\gamma_K=1}$. The differential of the Γ_K-action gives the Sen operator ∇_{Sen}. Since the Sen weights of t^iD_A are pointwisely $i \neq 0$, $\nabla_{\mathrm{Sen}} - i$ acts locally nilpotently on $D^\infty_{\mathrm{Sen}}(D_A)$. One gets that $D^\infty_{\mathrm{Sen}}(t^iD_A)^{\gamma_K=1} \subset D^\infty_{\mathrm{Sen}}(t^iD_A)^{\nabla_{\mathrm{Sen}}=0} = 0$. □

Glueing the construction in Lemma 3.10, we get a map

$$f_{\mathbf{h}} : (\mathfrak{X}_n)^{\wedge}_{\mathbf{h}} \to (\mathfrak{X}_n)^{\wedge}_0$$

and a commutative diagram of stacks over Rig_L

$$\begin{array}{ccccc} (\mathfrak{X}_n)^{\wedge}_{\mathbf{h}} & \xrightarrow{D_{\mathrm{pdR}}} & \widetilde{\mathfrak{g}}_{\mathbf{h}}/G & \longleftarrow & \widetilde{\mathfrak{g}}_{\mathbf{h}} \\ \downarrow f_{\mathbf{h}} & & \downarrow f_{\mathbf{h}} & & \downarrow f_{\mathbf{h}} \\ (\mathfrak{X}_n)^{\wedge}_0 & \xrightarrow{D_{\mathrm{pdR}}} & \mathfrak{g}/G & \longleftarrow & \mathfrak{g} \end{array}$$

where all squares are Cartesian by the following proposition.

Proposition 3.12. *The functor* $\Psi = (f_{\mathbf{h}}, D_{\mathrm{pdR}})$ *induces an equivalence*

$$\Psi : (\mathfrak{X}_n)^{\wedge}_{\mathbf{h}} \simeq (\mathfrak{X}_n)^{\wedge}_0 \times_{\mathfrak{g}/G} \widetilde{\mathfrak{g}}_{\mathbf{h}}/G$$

of stacks over Rig_L.

Proof. The 2-fiber product $(\mathfrak{X}_n)^{\wedge}_0 \times_{\mathfrak{g}/G} \widetilde{\mathfrak{g}}_{\mathbf{h}}/G$ is a stack [Sta24, Tag 026G] and for an affinoid $\mathrm{Sp}(A) \in \mathrm{Rig}_L$, the groupoid $((\mathfrak{X}_n)^{\wedge}_0 \times_{\mathfrak{g}/G} \widetilde{\mathfrak{g}}_{\mathbf{h}}/G)(A)$ is the category of tuples

$$(\Delta_A, D_{\mathrm{pdR},A}, \nu_A, \mathrm{Fil}^{\bullet} D_{\mathrm{pdR},A}, \alpha_A)$$

where Δ_A is a (φ, Γ_K)-module of rank n over $\mathcal{R}_{A,K}$ almost de Rham of weight 0, a triple

$$(D_{\mathrm{pdR},A}, \nu_A, \mathrm{Fil}^{\bullet} D_{\mathrm{pdR},A}) \in \widetilde{\mathfrak{g}}_{\mathbf{h}}/G(A)$$

and $\alpha_A : D_{\mathrm{pdR},A} \simeq D_{\mathrm{pdR}}(\Delta_A)$ is an isomorphism of $A \otimes_{\mathbb{Q}_p} K$-modules compatible with the nilpotent operators. Suppose that $g_1, g_2 \in \mathrm{Isom}(D_A, D'_A)$ for $D_A, D'_A \in (\mathfrak{X}_n)^{\wedge}_{\mathbf{h}}$. If $\Psi(g_1) = \Psi(g_2)$, then $g_1 = g_2 : D_A[\frac{1}{t}] = f_{\mathbf{h}}(D_A)[\frac{1}{t}] \to D'_A[\frac{1}{t}] = f_{\mathbf{h}}(D'_A)[\frac{1}{t}]$. Since D_A is t-torsion free, we see $g_1 = g_2$. Now suppose g is an isomorphism $(f_{\mathbf{h}}(D_A), \mathrm{Fil}^{\bullet} D_{\mathrm{pdR}}(D_A), \alpha_A) \simeq (f_{\mathbf{h}}(D'_A), \mathrm{Fil}^{\bullet} D_{\mathrm{pdR}}(D'_A), \alpha'_A)$ where α_A, α'_A are the natural identifications, e.g., $D_{\mathrm{pdR}}(D_A) = D_{\mathrm{pdR}}(D_A[\frac{1}{t}]) = D_{\mathrm{pdR}}(f_{\mathbf{h}}(D_A))$. By [KPX14, Lem. 2.2.9], the morphism $g : f_{\mathbf{h}}(D_A) \to f_{\mathbf{h}}(D'_A)$ is induced by some map $f_{\mathbf{h}}(D_A)^r \to f_{\mathbf{h}}(D'_A)^r$ uniquely determined by $f_{\mathbf{h}}(D_A)$ and $f_{\mathbf{h}}(D'_A)$ for some $r > 0$ such that $m(r)$ is large enough. By the uniqueness in the proof of Lemma 3.10, $f_{\mathbf{h}}(D_A)^r = f_{\mathbf{h}}(D^r_A)$ and $f_{\mathbf{h}}(D'_A)^r = f_{\mathbf{h}}((D'_A)^r)$. Then Proposition A.6 and Proposition A.3 implies that there exists $g' : D^r_A \simeq (D'_A)^r$ which induces g. This shows that Ψ is fully faithful by [Sta24, Tag 04WQ]. Also given $(\Delta_A, D_{\mathrm{pdR},A}, \nu_A, \mathrm{Fil}^{\bullet} D_{\mathrm{pdR},A}, \alpha_A)$, the triple $(D_{\mathrm{pdR},A}, \nu_A, \mathrm{Fil}^{\bullet} D_{\mathrm{pdR},A})$ and α_A define a lattice $D^{m,+}_{\mathrm{dif}}$ of $D^m_{\mathrm{dif}}(\Delta_A)$ by Proposition A.6 for $m = m(r)$ if Δ_A is the base change from a (φ, Γ_K)-module over $\mathcal{R}^r_{A,K}$ for some r such that $m(r)$ is large enough, which gives a modification D_A of Δ_A by Proposition A.3. This shows that the tuple lies in the essential image of Ψ_A. Then Ψ is an equivalence by [Sta24, Tag 046N]. ☐

Corollary 3.13. *The map* $f_{\mathbf{h}} : (\mathfrak{X}_n)^{\wedge}_{\mathbf{h}} \to (\mathfrak{X}_n)^{\wedge}_0$ *is projective, i.e., for any* $\mathrm{Sp}(A) \in \mathrm{Rig}_L$ *with* $\mathrm{Sp}(A) \to (\mathfrak{X}_n)^{\wedge}_0$*, the fiber product* $f_{\mathbf{h}}^{-1}(\mathrm{Sp}(A)) := \mathrm{Sp}(A) \times_{(\mathfrak{X}_n)^{\wedge}_0} (\mathfrak{X}_n)^{\wedge}_{\mathbf{h}}$ *is isomorphic to a rigid analytic space projective over* $\mathrm{Sp}(A)$.

Proof. By Proposition 3.12, $\mathrm{Sp}(A) \times_{(\mathfrak{X}_n)^{\wedge}_0} (\mathfrak{X}_n)^{\wedge}_{\mathbf{h}} = \mathrm{Sp}(A) \times_{\mathfrak{g}/G} \widetilde{\mathfrak{g}}_{\mathbf{h}}/G$. Let $(D_{\mathrm{pdR},A}, \nu_A)$ be the universal $A \otimes_{\mathbb{Q}_p} K$-module over A induced from $\mathrm{Sp}(A) \to \mathfrak{g}/G$. Then $f_{\mathbf{h}}^{-1}(\mathrm{Sp}(A))$ is the stack over $\mathrm{Rig}_L/\mathrm{Sp}(A)$ of ν_A-stable filtrations on $D_{\mathrm{pdR},A}$ of type $\mathbf{h}$. This stack is representable by the representability of Grassmannians and that being ν_A-stable is a Zariski-closed condition (essentially given by vanishing of matrix coefficients for morphisms between vector bundles). ☐

Remark 3.14. Locally, we can choose a trivialization $D_{\mathrm{pdR},A} \simeq (A \otimes_{\mathbb{Q}_p} K)^n$ in the above proof, equivalently choose a section $\mathrm{Sp}(A) \to \mathfrak{g}$ for $\mathrm{Sp}(A) \to \mathfrak{g}/G$. The map $\mathrm{Sp}(A) \to \mathfrak{g}$ is defined by ν_A with the set-theoretical image contained in the nilpotent cone $\mathcal{N}$, and $f_{\mathbf{h}}^{-1}(\mathrm{Sp}(A)) = \mathrm{Sp}(A) \times_{\mathfrak{g}} \widetilde{\mathfrak{g}}_{\mathbf{h}}$.

3.3. **Change of weights for general families.** We point out that change of weights for (φ, Γ_K)-modules may work for more general families, without pointwisely fixed Sen weights. Let D_A be a (φ, Γ_K)-module over $\mathcal{R}_{A,K}$ for an affinoid $\mathrm{Sp}(A) \in \mathrm{Rig}_L$. We fix $\sigma \in \Sigma$ and write $P_{\mathrm{Sen}}(T) \in (A \otimes_{\mathbb{Q}_p} K)[T]$ for the Sen polynomial of D_A, and $P_{\mathrm{Sen},\sigma}(T)$ for its σ-component via $A \otimes_{\mathbb{Q}_p} K \simeq \prod_\sigma A$. We will call $P_{\mathrm{Sen},\sigma}(T)$ the σ-Sen polynomial in the following.

Lemma 3.15. *Two polynomials* $Q(T)$ *and* $S(T)$ *in* $A[T]$ *are coprime to each other* ($(Q(T), S(T)) = (1)$) *if and only if for any* $x \in \mathrm{Sp}(A)$*, the sets of roots of* $Q(T) \otimes_A k(x)$ *and of* $S(T) \otimes_A k(x)$ *in* $k(x)$ *have empty intersection.*

Proof. The condition that $(Q(T), S(T)) = (1)$ is equivalent to that there is no maximal ideal $\mathfrak{m}$ of $A[T]$ containing both $S(T)$ and $P(T)$. Any maximal ideal $\mathfrak{m}$ of $A[T]$ lies over a point $x \in \mathrm{Sp}(A)$ [Sta24, Tag 00GB]. The result follows. ☐

Proposition 3.16. *Suppose that the* σ*-Sen polynomial* $P_{\mathrm{Sen},\sigma}(T)$ *of* D_A *admits a decomposition* $P_{\mathrm{Sen},\sigma}(T) = Q(T)S(T)$ *in* $A[T]$ *by monic polynomials such that* $(Q(T), S(T)) = (1)$*. Then there exists a unique* (φ, Γ_K)*-module* D'_A *over* $\mathcal{R}_A$ *contained in* D_A *and containing* tD_A *such that the Sen polynomial of* D'_A *is equal to* $Q(T-1)S(T)\prod_{\sigma' \neq \sigma} P_{\mathrm{Sen},\sigma}(T) \in \prod_{\sigma \in \Sigma} A[T]$.

Proof. We suppose that $D_A = D_A^r \otimes_{\mathcal{R}_{A,K}^r} \mathcal{R}_{A,K}$ for a (φ, Γ_K)-module D_A^r over $\mathcal{R}_{A,K}^r$. By Proposition A.3, it is enough to prove the following statement for a semilinear Γ_K-representation $D_{\mathrm{dif},A,\sigma}^{m,+}$ over $(A \otimes_{\sigma,K} K_m)[[t]]$: suppose that the characteristic polynomial for the Sen operator $\nabla = \nabla_{\mathrm{Sen}}$ on $D_{\mathrm{dif},A,\sigma}^{m,+}/t$ is equal to $P_{\mathrm{Sen},\sigma}(T) = Q(T)S(T)$ such that $(Q(T), S(T)) = (1)$, then there exists a unique sub-Γ_K-representation M contained in $D_{\mathrm{dif},A,\sigma}^{m,+}$ and containing $tD_{\mathrm{dif},A,\sigma}^{m,+}$ such that the Sen polynomial of M/t is equal to $Q(T-1)S(T)$.

Since $(Q(T), S(T)) = 1$, $A[T]/P_{\mathrm{Sen},\sigma}(T) = (A[T]/Q(T)) \times (A[T]/S(T))$ by the Chinese remainder theorem. As $P_{\mathrm{Sen},\sigma}(\nabla)$ annihilates $D_{\mathrm{dif},A,\sigma}^{m,+}/t$, this leads to a canonical decomposition $D_{\mathrm{dif},A,\sigma}^{m,+}/t = M_Q \oplus M_S$ where $Q(\nabla)$ kills M_Q and $S(\nabla)$ is invertible on M_Q. Since the actions of Γ_K, A and K_m commute with ∇, M_Q and M_S are Γ_K-stable projective $A \otimes_{\sigma,K} K_m$-modules.

We claim that M_Q is projective over $A \otimes_{\sigma,K} K_m$ of rank $\deg(Q)$ with characteristic polynomial of ∇ equaling to $Q(T)$. We can check the rank at points $\mathrm{Sp}(L') \to \mathrm{Spec}(A)$ for a finite extension L' over L and reduce to the case when $A = L'$ such that $\mathrm{Hom}_K(K_m, L') = [K_m : K]$. Then $D_{\mathrm{dif},L',\sigma}^{m,+}/t = \prod_{\sigma' \in \mathrm{Hom}_K(K_m,L')} D_{\mathrm{Sen},L',\sigma'}^m$ where $D_{\mathrm{Sen},L',\sigma'}^m = D_{\mathrm{Sen},L'}^m \otimes_{L' \otimes_{\sigma,K} K_m, 1\otimes\sigma'} L'$. And Γ_K permutes and induces ∇-equivariant isomorphisms between different σ'-factors. We have decompositions $D_{\mathrm{Sen},L',\sigma'}^m = M_{\sigma',Q} \oplus M_{\sigma',S}$ for all σ'. Up to enlarging L', the decomposition refines to a decomposition by generalized eigenspaces for ∇ whose dimensions are given by multiplicities of roots of $P_{\mathrm{Sen},\sigma}$. Then we see the rank over L' of each $M_{\sigma',Q}$ is equal to the degree of Q (and with the characteristic polynomial of ∇ equaling $Q(T)$). Hence $M_Q = \prod_{\sigma'} M_{\sigma',Q}$ is projective over $L' \otimes_{\sigma,K} K_m$ with the expected rank. Return to general A, let $Q'(T), S'(T)$ be the characteristic polynomials of ∇ on M_Q, M_S. Then $Q'(T)S'(T) = Q(S)S(T)$ and $(Q'(T), S'(T)) = (Q(T), S'(T)) = (Q'(T), S(T)) = 1$ by Lemma 3.15. We conclude that $A[T]/Q'(T) = A[T]/Q(T)$, hence $Q'(T) = Q(T)$ since both are monic polynomials of the same degree.

We take $M := \ker(D_{\mathrm{dif},A,\sigma}^{m,+} \to D_{\mathrm{dif},A,\sigma}^{m,+}/t \twoheadrightarrow M_Q)$. There is a Γ_K-filtration of $(A \otimes_{\sigma,K} K_m)[[t]]$-submodules

$$t^2 D_{\mathrm{dif},A,\sigma}^{m,+} \subset tM \subset tD_{\mathrm{dif},A,\sigma}^{m,+} \subset M \subset D_{\mathrm{dif},A,\sigma}^{m,+}$$

with graded pieces tM_S, tM_Q, M_S, M_Q. Then M/tM admits a Γ_K-filtration with graded pieces tM_Q and M_S. Since $\nabla(tx) = t(\nabla + 1)x$ for $x \in M_Q$, the characteristic polynomial of ∇ on M/tM is $Q(T-1)S(T)$. And M is finite projective over $(A \otimes_{\sigma,K} K_m)[[t]]$ by Lemma 3.17 below. The uniqueness comes from the uniqueness of the decomposition $M = M_Q \oplus M_S$. □

Lemma 3.17. *Let B be a Noetherian ring. Let M be a submodule of a finite projective $B[[t]]$-module D containing t^kD such that D/M is finite flat over $B[[t]]/t^k$. Then M is a finite projective $B[[t]]$-module of the same rank as D.*

Proof. Certainly M is finite over $B[[t]]$. Use the sequence $0 \to M \to D \to D/M \to 0$ and that D is flat over $B[[t]]$, we have $\mathrm{Tor}_i^{B[[t]]}(-, M) = \mathrm{Tor}_{i+1}^{B[[t]]}(-, D/M)$ for $i \geq 1$. For any $B[[t]]$-module N, there is a spectral sequence [Sta24, Tag 061Y]

$$\mathrm{Tor}_n^{B[[t]]/t^k}(\mathrm{Tor}_m^{B[[t]]}(N, B[[t]]/t^k), D/M) \Rightarrow \mathrm{Tor}_{n+m}^{B[[t]]}(N, D/M).$$

The flatness of D/M over $B[[t]]/t^k$ implies that $\mathrm{Tor}_n^{B[[t]]/t^k}(\mathrm{Tor}_m^{B[[t]]}(N, B[[t]]/t^k), D/M) = 0$ for $n \geq 1$. Thus the spectral sequence degenerates at the E_2-page and

$$\mathrm{Tor}_i^{B[[t]]}(-, D/M) = \mathrm{Tor}_i^{B[[t]]}(-, B[[t]]/t^k) \otimes_{B[[t]]/t^k} D/M$$

for all $i \geq 0$. Since $B[[t]]/t^k$ admits a flat resolution $0 \to t^kB[[t]] \to B[[t]] \to B[[t]]/t^k \to 0$, $\mathrm{Tor}_i^{B[[t]]}(-, B[[t]]/t^k) = 0$ for all $i \geq 2$. Hence $\mathrm{Tor}_i^{B[[t]]}(-, M) = 0$ for all $i \geq 1$. This implies that M is a finite flat $B[[t]]$-module. □

Example 3.18. Suppose that D_A has weights $\mathbf{h} \in (\mathbb{Z}^n)^\Sigma$ as in Lemma 3.10. Pick $\sigma \in \Sigma$ and assume that $\{h_{\sigma,1}, \cdots, h_{\sigma,n}\} = \{-k_1, \cdots, -k_s\}$ as sets where $-k_1 < \cdots < -k_s$ and each $-k_i$ appears m_i times in $\mathbf{h}_\sigma$. Let I be the nilradical of A. Then $P_{\mathrm{Sen},\sigma}(T) \equiv \prod_{i=1}^s (T + k_i)^{m_i} \mod I$. By Hensel's lemma [Sta24, Tag 0ALI], there exist coprime monic polynomials $Q(T), S(T)$ such that $P_{\mathrm{Sen},\sigma}(T) = Q(T)S(T)$ and $Q(T) \equiv (T + k_1)^{m_1} \mod I$. The above proposition gives a (φ, Γ_K)-module $D_A' \subset D_A$ such that $D_A'[\frac{1}{t}] = D_A[\frac{1}{t}]$ of σ-weights $h_{\sigma,1}+1 = \cdots = h_{\sigma,m_1}+1 \leq h_{\sigma,m_1+1} \leq \cdots$. Repeating such procedures for all σ and multiplying suitable powers of t, we can find in the end D_A' such that $D_A' = f_{\mathbf{h}}(D_A)$ is almost de Rham of weight 0 and $D_A'[\frac{1}{t}] = D_A[\frac{1}{t}]$.

3.4. **Flatness of the local model map.** We prove that the local model map $D_{\mathrm{pdR}} : (\mathfrak{X}_n)_0^\wedge \to \mathfrak{g}/G$ is flat in the sense of Corollary 3.28 and in the case $n = 2$, $K = \mathbb{Q}_p$. This part is to explain Hypothesis 5.9 that will appear in our main theorem (Theorem 5.15 and also Proposition 5.10). From now on we assume $n = 2$ and $K = \mathbb{Q}_p$. The major tool will be miracle flatness. Most proofs in this section work for more general situations, except for the very last part of the proof of Proposition 3.25.

We will define and study flatness via morphisms between complete local rings. We first recall the definition of versal rings of stacks assuming existence.

Definition 3.19. Let $\mathfrak{X}$ be a stack over Rig_L with a morphism $x : \mathrm{Sp}(L') \to \mathfrak{X}$ where L' is a finite extension of L. Then x corresponds to an object in $\mathfrak{X}(L')$ denoted by $D_{L'}$.

(1) We define $\mathcal{F}_{\mathfrak{X},x}$ to be the groupoid fibered in $\mathcal{C}_{L'}$ sending $A' \in \mathcal{C}_{L'}$ to pairs $(D_{A'} \in \mathfrak{X}(A'), \iota_{A'} : D_{A'} \otimes_{A'} L' \simeq D_{L'})$ where we write $D_{A'} \otimes_{A'} L'$ for the pullback of $D_{A'}$ in $\mathfrak{X}$ along $\mathrm{Sp}(A'/\mathfrak{m}_{A'}) \to \mathrm{Sp}(A')$. A morphism $(A', D_{A'}, \iota_{A'}) \to (A'', D_{A''}, \iota_{A''})$ in $\mathcal{F}_{\mathfrak{X},x}$ is a map $A' \to A''$ in $\mathcal{C}_{L'}$ together with an isomorphism $D_{L'} \otimes_{A'} A'' \simeq D_{A''}$ compatible with $\iota_{A'}$ and $\iota_{A''}$, cf. [Sta24, Tag 07XD].
(2) A formal object of $\mathfrak{X}$ is a complete Noetherian local ring $(R, \mathfrak{m}_R)$ with $L' = R/\mathfrak{m}_R$ finite over L and objects $D_{R/\mathfrak{m}_R^n} \in \mathfrak{X}(R/\mathfrak{m}_R^n)$ together with isomorphisms $D_{R/\mathfrak{m}_R^n} \otimes_{R/\mathfrak{m}_R^n} R/\mathfrak{m}_R^{n-1} \simeq D_{R/\mathfrak{m}_R^{n-1}}$ for all n [Sta24, Tag 07X3]. This formal object is versal at the map $x : \mathrm{Sp}(L') \to \mathfrak{X}$ corresponding to $D_{R/\mathfrak{m}_R} \in D(L')$ if the induced map $\mathrm{Spf}(R) \to \mathcal{F}_{\mathfrak{X},x}$ is formally smooth. In this case we say R a versal ring of $\mathfrak{X}$ at x.

Remark 3.20. Let A' be an Artin local L-algebra with residue field L' finite over L, then A' is an L'-algebra in a unique way such that $L' \to A' \to A'/\mathfrak{m}_{A'} = L'$ is the identity map (i.e., $A' \in \mathcal{C}_{L'}$) since L' is formally étale over L [Sta24, Tag 04G3]. If A is an affinoid algebra and $\mathfrak{m}$ is a maximal ideal of A with residue field L', then the completion $\widehat{A}_\mathfrak{m}$ is an L'-algebra and is a versal ring of A at the point $x : \mathrm{Sp}(L') \to \mathrm{Sp}(A)$ pro-representing $\mathcal{F}_{\mathrm{Sp}(A),x}$. If L'' is a finite extension of L' and we let $x'' : \mathrm{Sp}(L'') \to \mathrm{Sp}(L') \to \mathrm{Sp}(A)$, then $\mathcal{F}_{\mathrm{Sp}(A),x''}$ is pro-represented by the base change $\widehat{A}_\mathfrak{m} \otimes_{L'} L''$. Furthermore, if $\mathfrak{Y}^\wedge$ is the completion of $\mathfrak{Y} = \mathrm{Sp}(A)$ with respect to an ideal $I \subset A$ in the way of Example 3.1 with an L'-point $x : \mathrm{Sp}(L') \to \mathfrak{Y}_1 \to \mathfrak{Y}^\wedge$, then by definition $\mathcal{F}_{\mathfrak{Y},x} = \mathcal{F}_{\mathfrak{Y}^\wedge,x}$ are both pro-represented by $\widehat{A}_\mathfrak{m} = \widehat{A}_\mathfrak{m}^\wedge$.

We choose an L-point $x \in (\mathfrak{X}_2)_0^\wedge(L) \subset \mathfrak{X}_2(L)$ with Sen weights $(0,0)$ corresponding to a (φ,Γ)-module over $\mathcal{R}_L$. Let $X_{D_L} = \mathcal{F}_{\mathfrak{X}_2,x}$ be the deformation problem over $\mathcal{C}_L$ sending $A \in \mathcal{C}_L$ to the groupoid of pairs (D_A, ι_A) where D_A is a (φ,Γ)-module over $\mathcal{R}_A$ and $\iota_A : D_A/\mathfrak{m}_A \simeq D_L$. Let $x_{\mathrm{pdR}} = D_{\mathrm{pdR}}(D_L)$ be the image of x in $(\mathfrak{g}/G)(L)$ given by $(D_{\mathrm{pdR}}(D_L), \nu_L)$ and write $X_{x_{\mathrm{pdR}}}$ be the deformation problem of $(D_{\mathrm{pdR}}(D_L), \nu_L)$ over $\mathcal{C}_L$ [BHS19, §3.1].

Proposition 3.21. *If D_L is not a twist by a character of an extension of $t^{-1}\mathcal{R}_L(\epsilon)$ by $\mathcal{R}_L$ (written as $[\mathcal{R}_L - t^{-1}\mathcal{R}_L(\epsilon)]$), then $D_{\mathrm{pdR}} : X_{D_L} \to X_{x_{\mathrm{pdR}}}$ is formally smooth.*

Proof. To show formally smoothness, we need to show that for any surjction $A' \to A = A'/I$ in $\mathcal{C}_L$ such that $\mathfrak{m}_{A'}I = 0$, any deformation (D_A, ι) of D_L and deformation $(D_{\mathrm{pdR},A'}, \nu_{A'})$ with an isomorphism $(D_{\mathrm{pdR},A'}, \nu_{A'}) \otimes_{A'} A \simeq (D_{\mathrm{pdR}}(D_A), \nu_A)$, there exists $(D_{A'}, \iota_{A'}) \in X_{D_L}(A')$ such that there exists an isomorphism $D_{\mathrm{pdR}}(D_{A'}) \simeq D_{\mathrm{pdR},A'}$ compatible with $\nu_{A'}$ and induces the corresponding isomorphism modulo I, see [Sta24, Tag 06HF].

It's more convenient for us to use the language of B-pairs: the equivalence between (φ,Γ)-modules and B-pairs [Ber08a] and the equivalence between $X_{x_{\mathrm{pdR}}}$ and deformations of almost de Rham B_{dR}-representations [BHS19, Lem. 3.1.4]. We follow the proof and notation of [Nak14, Prop. 2.30]. Write $W = (W_e, W_{\mathrm{dR}}^+) = (W_e(D_L), W_{\mathrm{dR}}^+(D_L))$ and $W_A = (W_{e,A}, W_{\mathrm{dR,A}}^+) = (W_e(D_A), W_{\mathrm{dR}}^+(D_A))$. Write $\mathrm{End}(W) = W^\vee \otimes W$ where the tensor is in the category of B-pairs. Choose basis of $W_{e,A}$ and $W_{\mathrm{dR},A}^+$. Then W_A gives us 1-cocycles $\rho_e : \mathcal{G}_{\mathbb{Q}_p} \to \mathrm{GL}_2(B_e \otimes_{\mathbb{Q}_p} A)$, $\rho_{\mathrm{dR}} : \mathcal{G}_{\mathbb{Q}_p} \to \mathrm{GL}_2(B_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} A)$ and a matrix $P \in \mathrm{GL}_2(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A)$ such that $P\rho_e(g)g(P)^{-1} = \rho_{\mathrm{dR}}(g)$ for any $g \in \mathcal{G}_{\mathbb{Q}_p}$. Choose an L-linear section $s : A \to A'$ of $A' \to A$ which gives us lifts $\widetilde{\rho}_e := s \circ \rho_e, \widetilde{\rho}_{\mathrm{dR}} := s \circ \rho_{\mathrm{dR}}$ and $\widetilde{P}$. These elements defines 2-cocyles. For example, $c_{\mathrm{dR}}^2 \in I \otimes_L Z^2(\mathcal{G}_{\mathbb{Q}_p}, \mathrm{End}_{B_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} L}(W_{\mathrm{dR}}^+))$ is defined such that (use $\mathrm{End}_{B_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} A'}(W_{\mathrm{dR,A}}^+ \otimes_A A') \otimes_{A'} I = I \otimes_L \mathrm{End}_{B_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} L}(W_{\mathrm{dR}}^+)$)

$$c_{\mathrm{dR}}^2(g_1, g_2) = \widetilde{\rho}_{\mathrm{dR}}(g_1 g_2) g_1(\widetilde{\rho}_{\mathrm{dR}}(g_2))^{-1} \widetilde{\rho}_{\mathrm{dR}}(g_1)^{-1} - 1, \forall g_1, g_2 \in \mathcal{G}_{\mathbb{Q}_p}.$$

The vanishing of $H^2(\mathcal{G}_{\mathbb{Q}_p}, \mathrm{End}(W))$ (Lemma 3.24 below) implies that in the class of $(\widetilde{\rho}_e, \widetilde{\rho}_{\mathrm{dR}}, \widetilde{P})$ there exists always a lift $(\rho_{e,A'}, \rho_{\mathrm{dR},A'}, P_{A'})$ which defines a B-pair $W_{A'} = (W_{e,A'}, W^+_{\mathrm{dR},A'})$ over A' deforming W_A. Moreover, standard arguments show that the set of deformations is an affine space under $I \otimes_L H^1(\mathcal{G}_{\mathbb{Q}_p}, \mathrm{End}(W))$ (see the proof of [Nak14, Lem. 2.28]). For example, another lift $\rho'_{\mathrm{dR},A'}$ of ρ_{dR} defines a 1-cocycle in $I \otimes_L Z^1(\mathcal{G}_{\mathbb{Q}_p}, \mathrm{End}_{B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L}(W^+_{\mathrm{dR}}))$:

$$c^1_{\mathrm{dR}}(g) = \rho'_{\mathrm{dR},A'}(g)\rho_{\mathrm{dR},A'}(g)^{-1} - 1, \forall g \in \mathcal{G}_{\mathbb{Q}_p}.$$

Now we consider lifts of $W_{\mathrm{dR},A}$ to $W_{\mathrm{dR},A'}$. An easier argument shows that the set of deformations is parametrized by $I \otimes_L H^1(\mathcal{G}_{\mathbb{Q}_p}, \mathrm{End}_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L}(W_{\mathrm{dR}}))$. The map between lifts induced by D_{pdR} corresponds to the natural map $I \otimes_L H^1(\mathcal{G}_{\mathbb{Q}_p}, \mathrm{End}(W)) \to I \otimes_L H^1(\mathcal{G}_{\mathbb{Q}_p}, \mathrm{End}_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L}(W_{\mathrm{dR}}))$. We conclude that the existence of deformations to A' of W_A with given image under D_{pdR} is equivalent to that the map $H^1(\mathcal{G}_{\mathbb{Q}_p}, \mathrm{End}(W)) \to H^1(\mathcal{G}_{\mathbb{Q}_p}, \mathrm{End}_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L}(W_{\mathrm{dR}}))$ between tangent spaces is surjective (cf. [Sta24, Tag 0E3R]).

To see when $H^1(\mathcal{G}_{\mathbb{Q}_p}, \mathrm{End}(W)) \to H^1(\mathcal{G}_{\mathbb{Q}_p}, W_{\mathrm{dR}})$ is surjective, we go back to the language of (φ, Γ)-modules (we can also use B-quotients as in [Ked09]). Consider the long exact sequence (see Lemma 3.24 below)

$$\cdots \to H^1_{\varphi,\gamma}(\mathrm{End}_{\mathcal{R}_L}(D_L)) \to H^1_{\varphi,\gamma}(\mathrm{End}_{\mathcal{R}_L}(D_L)/t\mathrm{End}_{\mathcal{R}_L}(D_L)) \to H^2_{\varphi,\gamma}(t\mathrm{End}_{\mathcal{R}_L}(D_L)) \to 0.$$

Using Lemma 3.23 below, the map $H^1_{\varphi,\Gamma}(\mathrm{End}_{\mathcal{R}_L}(D_L)) \to H^1(\mathcal{G}_{\mathbb{Q}_p}, W_{\mathrm{dR}}(\mathrm{End}_{\mathcal{R}_L}(D_L)))$ is surjective if and only if $H^2_{\varphi,\gamma}(t\mathrm{End}_{\mathcal{R}_L}(D_L)) = 0$. By local Tate duality, $H^2_{\varphi,\gamma}(t\mathrm{End}_{\mathcal{R}_L}(D_L))^\vee = H^0_{\varphi,\gamma}(t^{-1}\mathrm{End}_{\mathcal{R}_L}(D_L)^\vee(\epsilon)) = H^0_{\varphi,\gamma}(t^{-1}D_L \otimes_{\mathcal{R}_L} D_L^\vee(\epsilon))$ which is non zero if and only if there exists a non-zero morphism $f : D_L \to D_L(\epsilon z^{-1})$ of (φ, Γ)-modules. Let $f \neq 0$ be such a map. The kernel and image of f are (φ, Γ)-modules [Ber08a, Prop. 1.1.1]. If D_L is irreducible, then we get an injection $D_L \hookrightarrow D_L(\epsilon z^{-1})$ which must be an isomorphism as both modules have weights zero (cf. Lemma 3.10). This is not possible considering φ-slopes (as $\epsilon z^{-1}(p) = p^{-1}$). Hence we may suppose that D_L is split trianguline, namely there exist smooth characters $\delta_1, \delta_2 : \mathbb{Q}_p^\times \to L^\times$ and a short exact sequence of (φ, Γ)-modules

$$0 \to \mathcal{R}_L(\delta_1) \to D_L \to \mathcal{R}_L(\delta_2) \to 0.$$

We may also suppose that $\mathcal{R}_L(\delta_1)$ is the rank one kernel of f. Then we get an injection $\mathcal{R}_L(\delta_2) \hookrightarrow D_L(\epsilon z^{-1})$. Hence $\delta_1 \epsilon z^{-1} = \delta_2$ since $\mathrm{Hom}_{\varphi,\gamma}(\mathcal{R}_L(\delta), \mathcal{R}_L(\delta')) = H^0_{\varphi,\gamma}(\mathcal{R}_L(\delta'\delta^{-1})) \neq 0$ for two smooth characters δ, δ' if and only if $\delta = \delta'$. Then under the assumption that D_L is not of this form, D_{pdR} is formally smooth at D_L. □

Remark 3.22. In the case that D_{pdR} is not smooth at D_L, $f_{\mathbf{h}}^{-1}(D_L)$ may contain non-smooth points, see Lemma 3.24 below.

Lemma 3.23. *Suppose that D_L is a (φ, Γ)-module of Hodge-Tate-Sen weights all 0. Then the map $H^1_{\varphi,\gamma}(D_L) \to H^1(\mathcal{G}_{\mathbb{Q}_p}, W_{\mathrm{dR}}(D_L))$ factors through $H^1_{\varphi,\gamma}(D_L/tD_L)$ and induces an isomorphism $H^1_{\varphi,\gamma}(D_L/tD_L) \simeq H^1(\mathcal{G}_{\mathbb{Q}_p}, W^+_{\mathrm{dR}}(D_L))$.*

Proof. Since $W^+_{\mathrm{dR}}(D_L)$ has weights 0, the map $H^1(\mathcal{G}_K, W^+_{\mathrm{dR}}(D_L)) \to H^1(\mathcal{G}_{\mathbb{Q}_p}, W_{\mathrm{dR}}(D_L))$ is an isomorphism. Moreover $H^1(\mathcal{G}_{\mathbb{Q}_p}, tW^+_{\mathrm{dR}}(D_L)) = 0$ is 0 (cf. [Nak14, Cor. 5.6]). We get isomorphisms

$$H^1(\mathcal{G}_{\mathbb{Q}_p}, W_{\mathrm{dR}}(D_L)) = H^1(\mathcal{G}_{\mathbb{Q}_p}, W^+_{\mathrm{dR}}(D_L)) = H^1(\mathcal{G}_{\mathbb{Q}_p}, W^+_{\mathrm{dR}}(D_L)/t).$$

There is a factorization $H^1_{\varphi,\gamma}(D_L) \to H^1(\mathcal{G}_{\mathbb{Q}_p}, W^+_{\mathrm{dR}}(D_L)) \to H^1(\mathcal{G}_{\mathbb{Q}_p}, W_{\mathrm{dR}}(D_L))$, see [Nak09, §2.1]. The cohomology of D_L and tD_L can be computed as $\mathcal{G}_{\mathbb{Q}_p}$-cohomology of complexes in the first two columns of the following short exact sequence of complexes of $\mathcal{G}_{\mathbb{Q}_p}$-modules (see *loc. cit.*):

$$\begin{array}{ccccc} W_e(tD_L) \oplus W^+_{\mathrm{dR}}(tD_L) & \longrightarrow & W_e(D_L) \oplus W^+_{\mathrm{dR}}(D_L) & \longrightarrow & W^+_{\mathrm{dR}}(D_L)/t \\ \downarrow & & \downarrow & & \downarrow \\ W_{\mathrm{dR}}(tD_L) & \longrightarrow & W_{\mathrm{dR}}(D_L) & \longrightarrow & 0. \end{array}$$

By comparing the long exact sequence for the cohomology of $0 \to tD_L \to D_L \to D_L/tD_L \to 0$ and using five lemma, we see $H^i_{\varphi,\gamma}(D_L/t) \simeq H^i(\mathcal{G}_{\mathbb{Q}_p}, W^+_{\mathrm{dR}}(D_L)/t)$ for $i = 0, 1$. □

Lemma 3.24. *If D_L has Hodge-Tate weights all 0, then $H^2_{\varphi,\gamma}(\mathrm{End}_{\mathcal{R}_L}(D_L)) = 0$ and X_{D_L} is formally smooth.*

Proof. By the local Tate duality,

$$H^2_{\varphi,\gamma}(\mathrm{End}_{\mathcal{R}_L}(D_L)) = H^0_{\varphi,\gamma}((D_L^\vee \otimes_{\mathcal{R}_L} D_L)^\vee(\epsilon))^\vee = \mathrm{Hom}_{\varphi,\gamma}(\mathcal{R}_L, D_L \otimes_{\mathcal{R}_L} D_L^\vee(\epsilon))^\vee.$$

Suppose that there is a non-zero map $g : \mathcal{R}_L \to D_L \otimes_{\mathcal{R}_L} D_L^\vee(\epsilon)$, which must be an injection and induces $B^+_{\mathrm{dR}} \hookrightarrow W^+_{\mathrm{dR}}(D_L \otimes_{\mathcal{R}_L} D_L^\vee(\epsilon))$. However, all Hodge-Tate-Sen weights of the (φ,Γ)-module $D_L \otimes_{\mathcal{R}_L} D_L^\vee(\epsilon)$ are equal to 1,

$$H^0(\mathcal{G}_{\mathbb{Q}_p}, W^+_{\mathrm{dR}}(D_L \otimes_{\mathcal{R}_L} D_L^\vee(\epsilon))) = \mathrm{Fil}^0 D_{\mathrm{dR}}(D_L \otimes_{\mathcal{R}_L} D_L^\vee(\epsilon)) = 0.$$

Hence $H^2_{\varphi,\gamma}(\mathrm{End}_{\mathcal{R}_L}(D_L)) = 0$. The formally smooth statement is [Che11, Prop. 3.6]. □

We choose a trivialization $\alpha_L : D_{\mathrm{pdR}}(D_L) \simeq L^2$ and let $X^\square_{x_{\mathrm{pdR}}}$ be the completion of $\mathfrak{g}$ at the corresponding matrix $\nu_L \in \mathfrak{g}$. Let $X^\square_{D_L} := X_{D_L} \times_{X_{x_{\mathrm{pdR}}}} X^\square_{x_{\mathrm{pdR}}}$, defined in [BHS19, §3.5]. Below, for a groupoid X fibered over $\mathcal{C}_L$, write $|X|$ for the corresponding functor taking isomorphism classes.

Proposition 3.25. *The morphism $X_{D_L} \to X_{x_{\mathrm{pdR}}}$ is relatively representable and is flat for all D_L in the sense of maps between versal rings.*

Proof. Let $\mathcal{M}_L = D_L[\frac{1}{t}]$ and let $X_{\mathcal{M}_L}$ be the groupoid of deformations of $\mathcal{M}_L$ in [BHS19, §3.3]. By [BHS19, Lem. 3.5.3], the map $X_{\mathcal{M}_L} \to X_{x_{\mathrm{pdR}}}$ is relatively representable. Let $W = (W_e, W^+_{\mathrm{dR}}) := (W_e(D_L), W^+_{\mathrm{dR}}(D_L))$ be the B-pair of D_L. We only need to show that the map $X_{D_L} \simeq X_{W_e} \times_{X_{W_{\mathrm{dR}}}} X_{W^+_{\mathrm{dR}}} \to X_{\mathcal{M}_L} \simeq X_{W_e}$ is relatively representable (we used [BHS19, Prop. 3.5.1]). We reduce to show that $X_{W^+_{\mathrm{dR}}} \to X_{W_{\mathrm{dR}}}$ is relatively representable. This is true and in our case $X_{W^+_{\mathrm{dR}}} \simeq X_{W_{\mathrm{dR}}}$ when W^+_{dR} has Hodge-Tate weight 0 (see Lemma 3.10 or [Wu21, Prop. 3.1]). Note that we get $X_{D_L} \simeq X_{W_e}$.

We may suppose that the map $X^\square_{D_L} = \mathrm{Spf}(R^\square_{D_L}) \to X^\square_{x_{\mathrm{pdR}}} = \mathrm{Spf}(S^\square)$ is induced by a continuous local morphism $S^\square \to R^\square_{D_L}$ of complete Noetherian local rings. We prove that this map is flat. By Proposition 3.21, this map is formally smooth, hence flat, if D_L is not an extension of $t^{-1}\mathcal{R}_L(\epsilon\delta)$ by $\mathcal{R}_L(\delta)$ for a character δ of $\mathbb{Q}_p^\times$. Otherwise, by miracle flatness [Sta24, Tag 00R3] and Lemma 3.24, it's enough to show that the fiber $R^\square_{D_L}/\mathfrak{m}_{S^\square}$ has codimension $\dim S^\square = 4$ in $R^\square_{D_L}$.

We first calculate the dimension of $R^\square_{D_L}$. Let $A = L[\epsilon] = L[\epsilon]/\epsilon^2$. The fibers of $|X^\square_{D_L}|(A) \to |X_{D_L}(A)|$ over given $(D_A, \iota_A) \in |X_{D_L}(A)|$ are isomorphisms $\alpha_A : D_{\mathrm{pdR}}(D_A) \simeq A^2$ parametrized by $\mathrm{End}_L(L^2)$. Two deformations given by α_A, α'_A are equivalent (give the same object in $|X^\square_{D_L}|(A)$) if and only if there exists an isomorphism $W_{e,A} \simeq W_{e,A}$ (in bijection with $H^0(\mathcal{G}_{\mathbb{Q}_p}, \mathrm{End}_{B_e \otimes_{\mathbb{Q}_p} L}(W_e))$) inducing $(\alpha')^{-1}\alpha$. The dimension of $|X_{D_L}(A)|$ is the dimension of $H^1(\mathcal{G}_{\mathbb{Q}_p}, \mathrm{End}_{B_e \otimes_{\mathbb{Q}_p} L}(W_e))$. The composite

$$\begin{aligned} H^0(\mathcal{G}_{\mathbb{Q}_p}, \mathrm{End}_{B_e \otimes_{\mathbb{Q}_p} L}(W_e)) &\to H^0(\mathcal{G}_{\mathbb{Q}_p}, \mathrm{End}_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L}(W_{\mathrm{dR}})) \\ &\simeq \mathrm{End}_{\mathrm{Rep}_L(\mathbf{G}_a)}(D_{\mathrm{pdR}}(W_{\mathrm{dR}})) \hookrightarrow \mathrm{End}_L(D_{\mathrm{pdR}}(W_{\mathrm{dR}})) \end{aligned}$$

is injective. Hence

$$\dim_L X^\square_{D_L}(A) = \dim_L H^1(\mathcal{G}_{\mathbb{Q}_p}, \mathrm{End}_{B_e \otimes_{\mathbb{Q}_p} L}(W_e)) + 4 - \dim_L H^0(\mathcal{G}_{\mathbb{Q}_p}, \mathrm{End}_{B_e \otimes_{\mathbb{Q}_p} L}(W_e)) = 8$$

by Euler characteristic formula and vanishing of H^2 (Lemma 3.24).

If $D_L = \mathcal{R}_L \oplus \mathcal{R}_L(t^{-1}\epsilon)$, $\nu_L = 0$, the quotient $R^\square/\mathfrak{m}_{S^\square}$ pro-represents the functor sending $A \in \mathcal{C}_L$ to the groupoid of (D_A, ι_A, α_A) where $\alpha_A : D_{\mathrm{pdR}}(D_A) \simeq A^n$ and D_A is de Rham. Since D_L is semi-stable, all its de Rham deformations are semi-stable by [Ber02, Thm. 0.9] and we have $D_{\mathrm{pdR}}(D_A) = D_{\mathrm{st}}(D_A)$ for such deformations. By the equivalence in [Ber08b], the category of semi-stable (φ,Γ)-modules of Hodge-Tate weights $(0,0)$ is equivalent to the category of (φ,N)-modules of rank 2. Hence the de Rham locus is the deformation space of matrices $(\varphi,N) \in \mathrm{GL}_2 \times \mathfrak{g}$ such that $N\varphi = p\varphi N$. This space has dimension 4 by [Hel23, Prop. 2.1].

If D_L is a non-split extension of $t^{-1}\mathcal{R}_L(\epsilon)$ by $\mathcal{R}_L$, then $\mathrm{End}_{\varphi,\gamma}(D_L) = L$ and the sheaf $|X_{D_L}|$ is pro-represented by a deformation ring R_{D_L} of dimension $\dim_L H^1_{\varphi,\gamma}(\mathrm{End}_{\mathcal{R}_L}(D_L)) = 5$ [Che11, Prop. 3.4] (even though X_{D_L} is not equivalent to $|X_{D_L}|$). The fiber product $X^\square_{D_L,\nu_L} := \mathrm{Spf}(R^\square_{D_L}) \times_{\mathrm{Spf}(S^\square)} \mathrm{Spf}(S^\square/\mathfrak{m}_{S^\square})$ pro-represents the groupoid fibered over $\mathcal{C}_L$ sending $A \in \mathcal{C}_L$ to the groupoid of (D_A, ι_A, α_A) where $\alpha_A : D_{\mathrm{pdR}}(D_A) \simeq A^n$ such that $\nu_A = \nu_L$ under the trivialization α_A. Let $X_{D_L,0}$ be the groupoid sending A to $(D_A, \iota_A) \in X_{D_L}(A)$ such that coefficients of Sen polynomials (given by $\mathrm{tr}(\nu_A)$ and $\det(\nu_A)$) of D_A vanish. Then $|X_{D_L,0}|$ is pro-represented by a quotient $R_{D_L,0}$ of R_{D_L}. Consider the map $|X^\square_{D_L,\nu_L}| \to |X_{D_L,0}|$. This map is formally smooth and of relative dimension 1 by Lemma 3.26 below. Thus $\dim_L R^{\square}_{D_L,\nu_L} = \dim_L R_{D_L,0} + 1$. We show that $\dim_L R_{D_L,0} = 3$, or has codimension 2 in R_{D_L}. Let $X^{z^{-1}\epsilon}_{D_L}$ be the deformation problem parametrizing deformations of D_L with fixed determinant $z^{-1}\epsilon$ (or $\mathcal{R}_A(z^{-1}\epsilon)$). Then $|X^{z^{-1}\epsilon}_{D_L}|$ is pro-represented by a complete Noetherian local ring $R^{z^{-1}\epsilon}_{D_L}$. Let $R_{\mathcal{R}_L}$ be the dimension 2 universal deformation ring of the trivial rank one (φ,Γ)-module. We have $R_{D_L} = R^{z^{-1}\epsilon}_{D_L}\widehat{\otimes}_L R_{\mathcal{R}_L}$ (see Lemma 3.27 below). Since fixed weight deformation of the trivial character has dimension one, it's enough to show that the fixed determinant deformation ring is flat over $L[[h]]$ where h is sent to the element in $R^{z^{-1}\epsilon}_{D_L}$ given by the trace of the universal nilpotent operator ν. By Krull's principal ideal theorem, a minimal prime of $R^{z^{-1}\epsilon}_{D_L}$ containing h has either height one or height zero. If all minimal primes containing h has height one, then $\dim R^{z^{-1}\epsilon}_{D_L}/h = \dim R^{z^{-1}\epsilon}_{D_L} - 1 = 2$ as desired. If the minimal prime containing h has height zero, since $R^{z^{-1}\epsilon}_{D_L}$ is integral (even regular, as is for R_{D_L}), we see $h = 0$ in $R^{z^{-1}\epsilon}_{D_L}$. This is not possible since we can construct trianguline deformations of the form $D_A = [\mathcal{R}_A(\delta_A^{-1}) - t^{-1}\mathcal{R}_A(\delta_A\epsilon)]$ for some $\delta_A : \mathbb{Q}_p^\times \to A^\times$ and $A = L[\epsilon]$ such that the weight of δ_A is not zero and D_A deforms D_L (the cokernel of $H^1_{\varphi,\gamma}(\mathcal{R}_A(z\delta_A^{-2}\epsilon^{-1})) \to H^1_{\varphi,\gamma}(\mathcal{R}_L(z\epsilon^{-1}))$ maps injectively into $H^2_{\varphi,\gamma}(\mathcal{R}_L(z\epsilon^{-1})) = 0$). □

Lemma 3.26. *The map $|X^{\square}_{D_L,\nu_L}| \to |X_{D_L,0}|$ in the above proof is formally smooth of relative dimension* 1*.*

Proof. Let $A' \to A = A'/I$ be a surjection in $\mathcal{C}_L$ such that $I^2 = 0$. Let $\nu_{A'} \in \mathrm{End}_{A'}(A'^2)$ such that $\nu_{A'} \equiv \nu_L = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}$ mod I and $\det(\nu_{A'}) = \mathrm{tr}(\nu_{A'}) = 0$. We claim that there exists $M \in I_2 + IM_2(A')$ (where I_2 is the identity matrix) such that $M\nu_{A'}M^{-1} = \nu_L$. To show the claim, write $\nu_{A'} = \begin{pmatrix} a & 1+b \\ c & d \end{pmatrix}$ for $a,b,c,d \in IA'$ and suppose that $M = I_2 + \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ for $x,y,z,w \in IA'$. Since $I^2 = 0$, one can calculate that $M\nu_{A'} = \nu_L M$ if and only if $\begin{pmatrix} z & w-x \\ & -z \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. As $\det(\nu_{A'}) = \mathrm{tr}(\nu_{A'}) = 0$ implies that $d = -a$ and $c = 0$, the solutions exist.

The above discussion shows that the map is formally smooth. To see the relative dimension, we only need to calculate the difference between tangent spaces. Let $A = L[\epsilon]$. The fiber in $|X^{\square}_{D_L,\nu_L}|(A)$ over given $(D_A, \iota_A) \in |X_{D_L,0}|(A)$ (by the above discussion we may suppose that $\nu_A = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}$) consists of matrices M of the form $I_2 + \epsilon\begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$ for $x, y \in L$ which span a space of dimension two. Two such matrices give the same object in $|X^{\square}_{D_L,\nu_L}|(A)$ if and only if there exists an automorphism $D_A \to D_A$ which reduces to identity modulo ϵ (determined by an element in $L = H^0_{\varphi,\gamma}(\mathrm{End}_{\mathcal{R}_L}(D_L))$) and induces the automorphism of $D_{\mathrm{pdR}}(D_A)$. Hence the fibers between tangent spaces have dimension one. □

Lemma 3.27. *Let $A \in \mathcal{C}_L$ and let $\delta_A : \mathbb{Q}_p^\times \to A^\times$ be a continuous character such that $\delta_A \equiv 1 \mod \mathfrak{m}_A$. Then $\delta_A^{\frac{1}{2}} : \mathbb{Q}_p^\times \to A^\times, x \mapsto \delta_A(x)^{\frac{1}{2}} := \sum_{i\geq 0}\binom{\frac{1}{2}}{i}(\delta_A(x)-1)^i$ defines the unique character such that $\delta_A^{\frac{1}{2}} \equiv 1 \mod \mathfrak{m}_A$ and $\delta_A = (\delta_A^{\frac{1}{2}})^2$.*

We will use the flatness in Proposition 3.25 in the following situation. Let $\mathfrak{Y} = \mathrm{Sp}(A)$ be an affinoid over L together with a (φ,Γ)-module D_A. Let $I \subset A$ be the ideal generalized by coefficients of the Sen polynomial cutting out the locus where D_A has Sen weights $(0,0)$ and $\mathfrak{Y}^\wedge$ the completion along this locus, $A^\wedge$ the completion with respect to I (cf. Appendix B). Let $h : \mathrm{Sp}(A) \to \mathfrak{X}_2$ and $h^\wedge : \mathfrak{Y}^\wedge \to (\mathfrak{X}_2)^\wedge_0$ be the morphism of stacks induced by D_A and $(D_{A/I^n})_n$. Let $D_{\mathrm{pdR}} : \mathfrak{Y}^\wedge \to (\mathfrak{X}_2)^\wedge_0 \overset{D_{\mathrm{pdR}}}{\to} \mathfrak{g}/G$ be the composite and let $D^{\square}_{\mathrm{pdR}} : \mathfrak{Y}^{\wedge,\square} := \mathfrak{Y}^\wedge \times_{\mathfrak{g}/G} \mathfrak{g} \to \mathfrak{g}$ be the base change.

Corollary 3.28. *In the above situation, suppose that h is smooth in the sense of versal maps: for any L'-point $y : \mathrm{Sp}(L') \to \mathrm{Sp}(A)$ and $x : \mathrm{Sp}(L') \overset{y}{\to} \mathrm{Sp}(A) \overset{h}{\to} \mathfrak{X}_2$, the induced map $\mathcal{F}_{\mathrm{Sp}(A),y} \to \mathcal{F}_{\mathfrak{X}_2,x} = X_{D_{L'}}$ is formally smooth. Then the map $D^{\square}_{\mathrm{pdR}} : \mathfrak{Y}^{\wedge,\square} \to \mathfrak{g}$ is flat in the sense that maps between versal rings at points with residue fields finite over L are flat.*

Proof. An L'-point $y^{\square}:\mathrm{Sp}(L')\to\mathfrak{Y}^{\wedge,\square}$ corresponds to a map $\mathrm{Sp}(L')\to A$ such that the pullback $D_{L'}$ has weights $(0,0)$ together with a framing $\alpha_{L'}: D_{\mathrm{pdR}}(D_{L'})\simeq(L')^2$. Let $\nu_{L'}\in\mathfrak{g}(L')$ be the nilpotent element. By the difinition of $\mathfrak{Y}^{\wedge,\square}$, the deformation problem $\mathcal{F}_{\mathfrak{Y}^{\wedge,\square},y^{\square}}$ sends $A'\in\mathcal{C}_{L'}$ to pairs $(\alpha_{A'}: D_{\mathrm{pdR}}(D_A\otimes_A A')\simeq(A')^2, A\to A')$ where $A\to A'$ is a morphism such that $A\to L'$ factors through A' and $\alpha_{A'}\equiv\alpha_{L'} \mod \mathfrak{m}_{A'}$. Then $(D_{A'}=D_A\otimes_A A',\iota_{A'}:D_{A'}\otimes_{A'} L'=D_{L'},\alpha_{A'})\in X^{\square}_{D_{L'}}(A')$ deforming $x^{\square}=(D_{L'},\iota_{L'},\alpha_{L'})\in X^{\square}_{D_{L'}}(L')$. By our assumption $\mathcal{F}_{\mathfrak{Y}^{\wedge,\square},y^{\square}}=\mathcal{F}_{\mathfrak{Y}^{\wedge},y}\times_{\mathcal{F}_{\mathfrak{g}/G,\nu_{L'}}}\mathcal{F}_{\mathfrak{g},\nu_{L'}}=\mathcal{F}_{\mathfrak{Y}^{\wedge},y}\times_{X_{D_{L'}}}X_{D^{\square}_{L'}}$ is formally smooth over $X_{D^{\square}_{L'}}$. By Proposition 3.25, the versal ring map which induces $X_{D^{\square}_{L'}}\to\mathcal{F}_{\mathfrak{g},\nu_{L'}}$ is flat. Hence the composite map $\mathcal{F}_{\mathfrak{Y}^{\wedge,\square},y^{\square}}\to\mathcal{F}_{\mathfrak{g},\nu_{L'}}$ is also flat. □

Remark 3.29. Locally on $\mathfrak{Y}_1$, one can choose a trivialization $D_{\mathrm{pdR}}(D_{A/I})$ on $\mathfrak{Y}_1$ where $\mathfrak{Y}_n=\mathrm{Sp}(A/I^n)$. Then $\mathfrak{Y}_1^{\square}=\mathfrak{Y}_1\times\mathrm{GL}_2$ and $D_{\mathrm{pdR}}(D_{\mathfrak{Y}_1})$ is free with a basis on $\mathfrak{Y}_1$. We can lift this basis by Nakayama lemma to a basis of $D_{\mathrm{pdR}}(D_{A^{\wedge}}):=\varprojlim_n D_{\mathrm{pdR}}(D_{A/I^n})$ which is a finite free $A^{\wedge}$-module by Lemma B.4. We get a trivialization $\mathfrak{Y}^{\wedge,\square}=\mathfrak{Y}^{\wedge}\times\mathrm{GL}_2$.

Cover $\mathfrak{Y}^{\wedge,\square}$ by affinoids of the form $\mathrm{Sp}(B/I)\times U$ where $\mathrm{Sp}(B)\subset\mathrm{Sp}(A)$ and $U\subset\mathrm{GL}_2$ are affinoid opens. Let $C=B\widehat{\otimes}_L\mathcal{O}(U)$ and $C^{\wedge}$ be its I-adic completion. The morphism $\mathrm{Sp}(C)^{\wedge}=\mathrm{Sp}(B)^{\wedge}\times U\to\mathfrak{g}$ of ringed sites factors through an affinoid $V\subset\mathfrak{g}$. In fact, write $\mathfrak{g}=\cup_{s\in\mathbb{N}}\mathfrak{g}_{\le p^s}:=\cup_{s\in\mathbb{N}}\mathrm{Sp}(L\langle p^s a,p^s b,p^s c,p^s d\rangle)$. The map $\mathrm{Sp}(B)^{\wedge}\times U\to\mathfrak{g}$ is determined by sending a,b,c,d to the matrix coefficients of $\nu_{C^{\wedge}}\in\mathrm{End}_{C^{\wedge}}(D_{\mathrm{pdR}}(D_{C^{\wedge}}))$ under the trivialization $D_{\mathrm{pdR}}(D_{C^{\wedge}})\simeq(C^{\wedge})^2$ on $\mathrm{Sp}(C)^{\wedge}$. Take s such that $p^s a,\cdots,p^s d$ are topologically nilpotent in C/I. Then for any $n\ge1$, $p^s a,\cdots,p^s d$ are also topologically nilpotent in C/I^n and induces $\varinjlim_n\mathrm{Sp}(C/I^n)\to\mathfrak{g}_{\le p^s}$. The ring maps $\mathcal{O}(\mathfrak{g}_{\le p^s})\to C^{\wedge}$ and $L[a,b,c,d]\to C^{\wedge}$ are flat by Corollary 3.28 and Lemma 3.30 below.

Lemma 3.30. *Let A be an Noetherian ring with an ideal I and I-adic completion $A^{\wedge}$. Let B be another Noetherian ring and $g:B\to A^{\wedge}$ be a morphism of rings. Suppose that for any maximal ideal $\mathfrak{m}$ of $A^{\wedge}$ and $\mathfrak{n}=g^{-1}(\mathfrak{m})$, the homomorphism $\widehat{B}_{\mathfrak{n}}\to\widehat{A}^{\wedge}_{\mathfrak{m}}$ of complete local rings is flat. Then the ring map g is flat itself.*

Proof. Since I is in the Jacobson radical of $A^{\wedge}$, maximal ideals of $A^{\wedge}$ is in bijection with maximal ideals of $A/I=A^{\wedge}/I$. By [Sta24, Tag 00HT], its enough to show that for all $\mathfrak{m},\mathfrak{n}$ as above, the map $B_{\mathfrak{n}}\to(A^{\wedge})_{\mathfrak{m}}$ is flat. Since both $A^{\wedge}$ and B are Noetherian, the maps $(A^{\wedge})_{\mathfrak{m}}\to\widehat{A}^{\wedge}_{\mathfrak{m}}$ and $B_{\mathfrak{n}}\to\widehat{B}_{\mathfrak{n}}$ are faithfully flat, the flatness of the map between Zariski local rings follows (by definition). □

4. Translations in family for $\mathrm{GL}_2(\mathbb{Q}_p)$

We will recall the definition of translations for (φ,Γ)-modules in [Din23] in the $\mathrm{GL}_2(\mathbb{Q}_p)$ case. We will need some explicit calculations for translations in families. It will be shown that the translation from regular weights to non-regular weights is the same as the change of weights of (φ,Γ)-modules. From now on, $K=\mathbb{Q}_p$.

The Lie algebra $\mathfrak{g}=\mathfrak{gl}_2$ is spanned by

$$a^+=\begin{pmatrix}1&\\&0\end{pmatrix}, a^-=\begin{pmatrix}0&\\&1\end{pmatrix}, u^+=\begin{pmatrix}0&1\\&0\end{pmatrix}, u^-=\begin{pmatrix}0&\\1&0\end{pmatrix}.$$

We write

$$\mathfrak{z}=a^++a^-=\begin{pmatrix}1&\\&1\end{pmatrix}, \mathfrak{h}=a^+-a^-=\begin{pmatrix}1&\\&-1\end{pmatrix}$$

The Casimir operator $\mathfrak{c}=\mathfrak{h}^2-2\mathfrak{h}+4u^+u^-\in U(\mathfrak{g})$.

Let A be an affinoid algebra over L. A (φ,Γ)-module D_A over $\mathcal{R}_A$ is a $P^+=\begin{pmatrix}\mathbb{Z}_p\setminus 0&\mathbb{Z}_p\\&1\end{pmatrix}$-module and is a $\mathfrak{p}^+$-module where $\mathfrak{p}^+=L[a^+,u^+]$ via the identifications

$$\varphi=\begin{pmatrix}p&\\&1\end{pmatrix}, \Gamma=\begin{pmatrix}\mathbb{Z}_p^{\times}&\\&1\end{pmatrix}, L[[X]]=L[[\begin{pmatrix}1&\mathbb{Z}_p\\&1\end{pmatrix}]].$$

The actions of u^+ on D_A is given by $v\mapsto\frac{d}{dz}(1+X)^z v|_{z=0}=\log(1+X)v=tv$ and a^+ by $\nabla=\nabla_{\mathrm{Sen}}$ by the lemma below.

Lemma 4.1. *Suppose that D_A^r is a (φ,Γ)-module over $\mathcal{R}_A^r$, then the map $v\mapsto\frac{d}{d\gamma}|_{\gamma=1}\gamma.v,\gamma\in\mathbb{Z}_p^{\times}\simeq\Gamma$ defines the Sen operator ∇_{Sen} acting on D_A^r.*

Proof. The proof is the same as the case when A is a field using [Ber02, Lem. 5.2]. It is proved in [KPX14, Prop. 2.2.14] that the action of Γ on D_A^r extends to an action of the distribution algebra of Γ which contains the element a^+. □

Definition 4.2. A $(\varphi,\Gamma,\mathfrak{g})$-module over $\mathcal{R}_A^r$ is a (φ,Γ)-module D_A^r (in the sense of [KPX14, Def. 2.2.12]) over $\mathcal{R}_A^r$ with an A-linear continuous action of $U(\mathfrak{g})$ extending the action of $U(\mathfrak{p}^+)$ such that the $U(\mathfrak{g})$-action extends continuously to D_A^s for all $0<s<r$ and $\varphi u^- = p^{-1}u^-\varphi, \varphi a^- = a^-\varphi$ under $\varphi: D_A^r \to D_A^{r/p}$. A $(\varphi,\Gamma,\mathfrak{g})$-module over $\mathcal{R}_A$ is a (φ,Γ)-module over $\mathcal{R}_A$ with an action of $U(\mathfrak{g})$ that is the base change from $\mathcal{R}_A^r$ to $\mathcal{R}_A$ for a $(\varphi,\Gamma,\mathfrak{g})$-module over $\mathcal{R}_A^r$ and some $r>0$.

A $(\varphi,\Gamma,\mathfrak{g})$-module is naturally a $(P^+,\mathfrak{g})$-module defined in a similar way where P^+ acts continuously. If D_A is a $(\varphi,\Gamma,\mathfrak{g})$-module, then the action of $Z(\mathfrak{g})$ commutes with $\varphi,\Gamma,\mathcal{R}_A^r$.

Lemma 4.3. *Let D_A^r be a (φ,Γ)-module over $\mathcal{R}_A^r$ of rank two. Write $P_{\mathrm{Sen}}(T) = T^2-\gamma_1 T+\gamma_0 \in A[T]$ for the Sen polynomial of D_A. There exists a unique A-linear $\mathfrak{g}$-module structure on D_A^r such that $\mathfrak{c}$ acts on D_A^r by $(\gamma_1^2-4\gamma_0-1)$ and $\mathfrak{z}$ acts by γ_1-1 making D_A^r (resp. D_A) a $(\varphi,\Gamma,\mathfrak{g})$-module.*

Proof. This is just [Col18, Prop. 2.2]. We declare the (unique) action of a^- by $\mathfrak{z}-a^+$, $\mathfrak{h} = a^+-a^- = 2\nabla-\mathfrak{z} = 2\nabla-\gamma_1+1$ and u^- by $\frac{\mathfrak{c}-\mathfrak{h}^2+2\mathfrak{h}}{4u^+} = -\frac{P_{\mathrm{Sen}}(\nabla)}{t}$. The last one is possible because $P_{\mathrm{Sen}}(\nabla)D_A \subset tD_A$ by the same reason for [Col18, Lem. 1.6] using Proposition A.3. We can check this formally defines an action of $\mathfrak{g}$. For example for $v\in D_A$, we have $[u^+,u^-]v = (u^+u^- - u^-u^+)v = -P_{\mathrm{Sen}}(\nabla) + t^{-1}P_{\mathrm{Sen}}(\nabla)tv = -(\nabla^2-\gamma_1\nabla+\gamma_0)v + t^{-1}(\nabla^2-\gamma_1\nabla+\gamma_0)tv = (2\nabla-\gamma_1+1)v = \mathfrak{h}.v$ using that $\nabla(tv) = tv + t\nabla(v)$. We also check that $\varphi u^- = p^{-1}u^-\varphi$ using that φ commutes with $\mathfrak{z},\mathfrak{c}$ and $\varphi u^+ = pu^+\varphi$. □

Definition 4.4. We say the $\mathfrak{g}$-module structure in Lemma 4.3 the standard $\mathfrak{g}$-module structure for a rank 2 (φ,Γ)-module over $\mathcal{R}_A$ or $\mathcal{R}_A^r$.

Remark 4.5. It is possible to equip a (φ,Γ)-module with different $\mathfrak{g}$-structures. See [Din23, Rem. 2.14] for more discussions.

Let $V_k = \mathrm{Sym}^k L^2 = \mathcal{R}_L^+/X^{k+1}$ be the irreducible representation of $\mathfrak{g}$ of highest weight $(k,0)$ (for the Borel subalgebra the algebra of upper-triangular matrices). Here $\mathcal{R}_L^+ = D(\mathbb{Z}_p,L)\subset\mathcal{R}_L$ is the distribution algebra of $\mathbb{Z}_p$ identified with rigid analytic functions in variable X on the open unit disc. If D_A^r is a $(\varphi,\Gamma,\mathfrak{g})$-module, then $D_A^r\otimes_L V_k$ is also a $(P^+,\mathfrak{g})$-module via the diagonal action of P^+. The following observation is due to Ding.

Proposition 4.6 ([Din23, Prop. 2.1])**.** *Suppose that D_A is a $(\varphi,\Gamma,\mathfrak{g})$-module over $\mathcal{R}_A$. The diagonal action of $\mathcal{R}_A^+ = D(\mathbb{Z}_p,L)\widehat{\otimes}_L A$ extends to an action of $\mathcal{R}_A$ making $D_A\otimes_L V_k$ a $(\varphi,\Gamma,\mathfrak{g})$-module over $\mathcal{R}_A$. And there is a filtration*

$$0\subset D_A\otimes_L X^k\mathcal{R}_L^+/X^{k+1}\subset\cdots\subset D_A\otimes_L X^i\mathcal{R}_L^+/X^{k+1}\subset\cdots\subset D_A\otimes_L \mathcal{R}_L^+/X^{k+1}$$

of (φ,Γ)-modules (as well as $\mathfrak{p}^+$-modules, but not as $\mathfrak{g}$-modules) with graded pieces

$$D_A\otimes_L (X^i\mathcal{R}_L^+/X^{i+1})\simeq t^iD_A$$

for $0\le i\le k$.

Proof. By definition, there exists r such that D_A is the base change of D_A^r from a finite projective module over $\mathcal{R}_A^r$. We will prove (and will use) the statement for D_A^r.

The same proof of *loc. cit.* shows that the action of $\mathcal{R}_A^+$ extends to an action of $\mathcal{R}_A^r$ on $D_A^r\otimes_L V_k$. We give a direct proof here. Notice that for $X = [1]-1\in L[[\mathbb{Z}_p]]\subset D(\mathbb{Z}_p,L)$, $g\in D_A^r$ and $v\in V_k$, we have $X(X^{-1}g\otimes v) = g\otimes v + X^{-1}g\otimes Xv + g\otimes Xv$. We then get $\frac{1}{X}(g\otimes v) = X^{-1}g\otimes v - \frac{1}{X}(\frac{X+1}{X}g\otimes Xv) = X^{-1}g\otimes v - \frac{X+1}{X^2}g\otimes Xv + \frac{1}{X}((\frac{X+1}{X})^2g\otimes X^2v) = \sum_{i=0}^k(-\frac{X+1}{X})^iX^{-1}g\otimes X^iv$. Hence $X^{-m}(g\otimes v) = \sum_{i=0}^k\binom{m+i-1}{m-1}(-\frac{X+1}{X})^iX^{-m}g\otimes X^iv$. And we conclude that $f(X)(g\otimes v) = \sum_{i=0}^k\frac{1}{i!}(X+1)^if^{(i)}(X)g\otimes X^iv$ where $f^{(i)}$ denotes the i-th derivative of f. The extension of the action follows since if $f\in\mathcal{R}_A^r = \mathcal{O}(\mathbb{U}^r\times\mathrm{Sp}(A))$, so is its derivative for X. Then $D_A^r\otimes_L V_k$ becomes a projective $\mathcal{R}_A^r$-module with a $\mathcal{R}_A^r$-filtration. The projectivity is due to that extensions of projective modules are projective.

We extend the φ,Γ actions diagonally. For example $\varphi: D_A^r\to\mathcal{R}_A^{r/p}\otimes_{\mathcal{R}^r}D_A^r$ gives a map $D_A^r\otimes_L\mathcal{R}_L^+/X^{k+1}\to(\mathcal{R}_A^{r/p}\otimes_{\mathcal{R}_A^r}D_A^r)\otimes_L\mathcal{R}_L^+/X^{k+1}$. There is an $\mathcal{R}_A^{r/p}$-isomorphism $\Psi:\mathcal{R}_A^{r/p}\otimes_{\mathcal{R}_A^r}(D_A^r\otimes_L\mathcal{R}_L^+/X^{k+1})\simeq(\mathcal{R}_A^{r/p}\otimes_{\mathcal{R}_A^r}D_A^r)\otimes_L\mathcal{R}_L^+/X^{k+1}: f(X)\otimes(g\otimes v)\mapsto\sum_{i=0}^k(\frac{1}{i!}(X+1)^if^{(i)}(X)\otimes g)\otimes X^iv$

for $f(X)\in\mathcal{R}_A^{r/p}, g\in D_A^r, v\in\mathcal{R}_L^+/X^{k+1}$ which is well-defined and extends the $\mathcal{R}_L^+$-actions (given by the diagonal action of $\mathbb{Z}_p$). We get the desired φ-action map: $D_A^r\otimes_L\mathcal{R}_L^+/X^{k+1}\to\mathcal{R}_A^{r/p}\otimes_{\mathcal{R}_A^r}(D_A^r\otimes_L\mathcal{R}_L^+/X^{k+1}): g\otimes m\to\Psi^{-1}(\varphi(g)\otimes\varphi(m))$. The equality $\Psi^{-1}(\varphi(f(X+1).(g\otimes m)))=f((X+1)^p).(\Psi^{-1}(\varphi(g)\otimes\varphi(m)))$ is equivalent to that $\varphi(f(X+1).(g\otimes m))=f((X+1)^p).(\varphi(g)\otimes\varphi(m))$ which can be checked formally.

See Lemma 4.15 below for the last isomorphism. □

Lemma 4.7. *If $A\to B$ is a morphism of affinoid algebras and D_A is an $(\varphi,\Gamma,\mathfrak{g})$-module, then $D_A\otimes_{\mathcal{R}_A}\mathcal{R}_B$ is also a $(\varphi,\Gamma,\mathfrak{g})$-module and there is an isomorphism $(D_A\otimes_L\mathcal{R}_L^+/X^{k+1})\otimes_{\mathcal{R}_A}\mathcal{R}_B\simeq((D_A\otimes_{\mathcal{R}_A}\mathcal{R}_B)\otimes_L\mathcal{R}_L^+/X^{k+1})$ as $(\varphi,\Gamma,\mathfrak{g})$-modules.*

Proof. We let $Z(\mathfrak{g})\subset U(\mathfrak{g})$ acts on the (φ,Γ)-module $D_B:=D_A\otimes_{\mathcal{R}_A}\mathcal{R}_B$ by extending the scalars. To show that D_B is a $(\varphi,\Gamma,\mathfrak{g})$-module, we need to show that the action of $u^-=-\frac{1}{t}(\mathfrak{c}^2-\mathfrak{h}^2+2\mathfrak{h})$ is defined, or that $(\mathfrak{c}^2-(2\nabla-\mathfrak{z})^2+2(2\nabla-\mathfrak{z}))D_B\subset tD_B$. Using Proposition A.3, that $D_{\mathrm{Sen}}^m(D_B)=D_{\mathrm{Sen}}^m(D_A)\otimes_A B$, the extension of the u^- action follows. The map $(D_A\otimes_L\mathcal{R}_L^+/X^{k+1})\otimes_{\mathcal{R}_A}\mathcal{R}_B\to((D_A\otimes_{\mathcal{R}_A}\mathcal{R}_B)\otimes_L\mathcal{R}_L^+/X^{k+1})$ is given by $(g\otimes v)\otimes f\mapsto\sum_{i=0}^k\frac{1}{i!}(X+1)^if^{(i)}(X)g\otimes X^iv$ for $g\in D_A, v\in\mathcal{R}_L^+/X^{k+1}$. The map induces isomorphisms on graded pieces of the filtration in Proposition 4.6, hence is an isomorphism of (φ,Γ)-modules over $\mathcal{R}_B$ itself. It is a $\mathfrak{g}$-isomorphism since it is moreover compatible with the $\mathcal{R}_B$-linear actions of $Z(\mathfrak{g})$. □

We need recall translations of $\mathfrak{g}$-modules.

Lemma 4.8. *Let R be a ring.*

(1) *Suppose that R is commutative with a nilpotent ideal I and V_R is an R-module equipped with an R-linear endomorphism C. Let $a,b\in R$ such that $a-b\in I$. Then $V_R\{C=a\}=V_R\{C=b\}$ where $\{C=-\}$ denotes the generalized eigenspace.*

(2) *Let R be an L-algebra. Suppose that V_R is an R-module and Z is a commutative algebra finitely generated over L with an L-morphism $Z\to\mathrm{End}_R(V_R)$. Suppose that V_R is locally Z-finite (for any $v\in V_R$, $Z.v\subset V_R$ is finite-dimensional over L). Then the R-morphism $\oplus_{\mathfrak{m}\in\mathrm{SpecMax}(Z)}V_R[\mathfrak{m}^\infty]\to V_R$ is an isomorphism.*

(3) *Let $R\to S$ be a morphism of L-algebras. Let $V_S=V_R\otimes_R S$ and let Z act on V_S by extending the scalar. Suppose that V_R is locally Z-finite. Then for any $\mathfrak{m}\in\mathrm{SpecMax}(Z)$, the natural map $V_R[\mathfrak{m}^\infty]\otimes_R S\to V_S[\mathfrak{m}^\infty]$ is an isomorphism.*

Proof. (1) Let $x=b-a\in I$. Let $T=C-a$. Then $(T-x)^s=\sum_{i=0}^s\binom{s}{i}(-1)^{n-i}T^ix^{n-i}$. If $v\in V$ is killed by some power of T, since it is killed by some power of x, it is killed by some power $(T-x)^s=(C-b)^s$.

(2) The decomposition (as L-spaces) holds for all finite-dimensional Z-modules.

(3) This follows from the decompositions in (2) for V_R and V_S and that $V_R[\mathfrak{m}^\infty]\otimes_R S$ is mapped into $(V_R\otimes_R S)[\mathfrak{m}^\infty]$. □

For two weights $\lambda,\mu\in\mathfrak{t}^*=\mathrm{Hom}_L(\mathfrak{t},L)$ such that $\nu=\lambda-\mu\in\mathbb{Z}^2$ is integral, write $\overline{\nu}$ for the dominant weight in the Weyl group orbits of ν for the linear Weyl group action. We recall the following definitions (see [JLS21, §2.3, §2.4.1]). For $\lambda\in\mathfrak{t}^*$, let $\mathfrak{m}_\lambda\subset Z(\mathfrak{g})$ be the kernel of the infinitesimal character χ_λ attached to λ via the Harish-Chandra isomorphism.

Definition 4.9. Let M be a $U(\mathfrak{g})$-module.

(1) We say M is locally $Z(\mathfrak{g})$-finite if for any $v\in M$, the subspace $Z(\mathfrak{g}).v$ is finite-dimensional over L.
(2) If M is locally $Z(\mathfrak{g})$-finite, write $\mathrm{pr}_{|\lambda|}M:=M[\mathfrak{m}_\lambda^\infty]$.
(3) If M is locally $Z(\mathfrak{g})$-finite, $T_\mu^\lambda M:=\mathrm{pr}_{|\lambda|}(\mathrm{pr}_{|\mu|}M\otimes_L L(\overline{\nu}))$.

Remark that if M is $Z(\mathfrak{g})$-finite, then $M\otimes_L L(\overline{\nu})$ is also locally $Z(\mathfrak{g})$-finite (see [BG80, Cor. 2.6 (ii)]). Moreover, there is a direct sum decomposition $M=\oplus_{\mathfrak{m}\in\mathrm{SpecMax}(Z(\mathfrak{g}))}M[\mathfrak{m}^\infty]$. If $M=M[\mathfrak{m}^\infty]$ for some $\mathfrak{m}=\mathfrak{m}_\lambda$ and $\lambda+\rho$ is dominant, then

$$M\otimes V=\oplus_{\mu\in\mathrm{wt(V)}}\mathrm{pr}_{|\lambda+\mu|}(M\otimes V)$$

for any finite dimensional $\mathfrak{g}$-module V where μ runs over all $\mathfrak{t}$-weights appeared in V (see the end of [BG80, §2]). Moreover, if $0\to M_1\to M\to M_2\to 0$ is a short exact sequence of $U(\mathfrak{g})$-modules such that M_1,M_2 are locally $Z(\mathfrak{g})$-finite, then so is M.

Lemma 4.10. *Let $\lambda_1, \cdots, \lambda_n \in \mathfrak{t}^*$ such that $\lambda_{i+1} - \lambda_i \in X^*(\mathfrak{t})_+$ are dominant integral weights for all $n-1 \geq i \geq 1$ and suppose that $\lambda_1 + (1,0)$ is dominant. Then for a locally $Z(\mathfrak{g})$-finite $\mathfrak{g}$-module M, there are natural isomorphisms*

$$T_{\lambda_{n-1}}^{\lambda_n} \cdots T_{\lambda_1}^{\lambda_2} M \simeq T_{\lambda_1}^{\lambda_n} M \simeq \mathrm{pr}_{|\lambda_n|}(\mathrm{pr}_{|\lambda_1|} M \otimes L(\lambda_1 - \lambda_2) \otimes \cdots \otimes L(\lambda_n - \lambda_{n-1})).$$

Proof. The tensor product $L(\lambda_1 - \lambda_2) \otimes \cdots \otimes L(\lambda_n - \lambda_{n-1})$ has a direct summand $L(\lambda_n - \lambda_1)$ of multiplicity one and all the other summands are irreducible representations with highest weight $\mu < \lambda_n - \lambda_1$ by induction using Steinberg's theorem [Ste61]. □

Lemma 4.11. *Let D_A be a $(\varphi, \Gamma, \mathfrak{g})$-module over $\mathcal{R}_A$ that is locally $Z(\mathfrak{g})$-finite. Let I be the nilradical of A.*

(1) Suppose that $\chi_{1,A}, \chi_{2,A} : Z(\mathfrak{g}) \to A$ are two characters such that $\chi_1 \equiv \chi_2 \mod I$. Then $D_A\{Z(\mathfrak{g}) = \chi_{1,A}\} = D_A\{Z(\mathfrak{g}) = \chi_{2,A}\}$.

(2) Let $\lambda \in \mathfrak{t}^*$ and $\chi_A = \chi_\lambda$ corresponding to $\mathfrak{m}_\lambda \in \mathrm{SpecMax}(Z(\mathfrak{g}))$. Then the generalized eigenspace $D_A\{Z(\mathfrak{g}) = \chi_A\} = D_A[\mathfrak{m}_\lambda^\infty]$ is a (φ, Γ)-module.

(3) In the situation of (2). The functor $D_A\{Z(\mathfrak{g}) = \chi_\lambda\}$ is exact on locally $Z(\mathfrak{g})$-finite $(\varphi, \Gamma, \mathfrak{g})$-modules and its formation commutes with arbitrary base change, i.e., for any map $A \to B$ of L-affinoid algebras, we have $D_A[\mathfrak{m}_\lambda^\infty] \otimes_{\mathcal{R}_A} \mathcal{R}_B \simeq D_B[\mathfrak{m}_\lambda^\infty]$ where $D_B = D_A \otimes_{\mathcal{R}_A} \mathcal{R}_B$.

Proof. (1) An affinoid algebra A over L is Jacobson and Noetherian. The nilradical I is nilpotent. The statement follows from that $Z(\mathfrak{g})$ is finitely generated and Lemma 4.8.

(2) Since the action of $Z(\mathfrak{g})$ commutes with the $(\varphi, \Gamma, \mathfrak{g})$-module structure on D_A, we see $D_A\{Z(\mathfrak{g}) = \chi_A\}$ is a $\mathcal{R}_A$-module with compatible actions of $\varphi, \Gamma, \mathfrak{g}$. It suffices to show that $D_A\{Z(\mathfrak{g}) = \chi_A\}$ is a $(\varphi, \Gamma, \mathfrak{g})$-module. By (2) of Lemma 4.8, $D_A^r[\mathfrak{m}_\lambda^\infty]$ is a direct summand of D_A^r. Then $D_A^r[\mathfrak{m}_\lambda^\infty]$ is finite projective over $\mathcal{R}_A^r$ and we have $D_A\{Z(\mathfrak{g}) = \chi_A\} = \mathcal{R}_A \otimes_{\mathcal{R}_A^r} D_A^r\{Z(\mathfrak{g}) = \chi_A\}$.

(3) This also follows from Lemma 4.8. □

For Sen weights $\mathbf{h} = (h_1, h_2) \in \mathbb{Z}^2, h_1 \leq h_2$, we write $\lambda = \lambda_{\mathbf{h}} = (h_2 - 1, h_1)$ for the corresponding character of $\mathfrak{b}$ where $\lambda + (1,0)$ is dominant. Then χ_λ is the infinitesimal character $Z(\mathfrak{g}) \to L : \mathfrak{z} \mapsto h_1 + h_2 - 1, \mathfrak{c} \mapsto (h_1 - h_2)^2 - 1$.

Proposition 4.12. *Let D_A be an almost de Rham (φ, Γ)-module of rank two over $\mathcal{R}_A$ with Sen weights $(h_1, h_2), h_1 \leq h_2$, equipped with the standard $U(\mathfrak{g})$-module structure by Lemma 4.3. Then $D_A = D_A\{Z(\mathfrak{g}) = \chi_\lambda\}$. And for any $\mu \in X^*(\mathfrak{t})$, $T_\lambda^\mu D_A$ is an almost de Rham (φ, Γ)-module over $\mathcal{R}_A$ provided that $T_\lambda^\mu D_A \neq 0$. Moreover, the formation of $T_\lambda^\mu D_A$ commutes with base change.*

Proof. In Lemma 4.3, $Z(\mathfrak{g})$ act on D_A by a character $\chi_A(\mathfrak{z}) = \gamma_1 - 1$ and $\chi_A(\mathfrak{c}) = (\gamma_1^2 - 4\gamma_0 - 1)$ if the Sen polynomial equals to $T^2 - \gamma_1 T + \gamma_0$. Since D_A pointwisely has Sen weights h_1, h_2, the Sen polynomial equals to $(T - h_1)(T - h_2)$ at all closed points of $\mathrm{Spec}(A)$. An element $f \in A$ is in the nilradical I of A if $f \in \mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of A. We see $X^2 - \gamma_1 X + \gamma_0 \equiv X^2 - (h_2 + h_1)X + h_1 h_2 \mod I$. Thus $\chi_A \equiv \chi_\lambda \mod I$. Hence $D_A = D_A[Z(\mathfrak{g}) = \chi_A] = D_A\{Z(\mathfrak{g}) = \chi_\lambda\}$ by (1) of Lemma 4.11. The other statements follow from the same lemma, Lemma 4.7 and Proposition 4.6. □

Corollary 4.13. *We equip rank two (φ, Γ)-modules D_A with the standard $\mathfrak{g}$-structures.*

(1) Suppose that D_A is almost de Rham with pointwisely regular Sen weights $(h_1, h_2) \in \mathbb{Z}^2, h_1 < h_2$. Then for any integral weight $\mu = (h'_2 - 1, h'_1)$ such that $h'_1 < h'_2$, $T_\lambda^\mu D_A$ is almost de Rham of rank two pointwisely with Sen weights (h'_1, h'_2).

(2) Suppose that D_A is almost de Rham with pointwisely non-regular Sen weights $(h'_1, h'_2) \in \mathbb{Z}^2, \mu = (h'_2 - 1, h'_1), h'_1 = h'_2$. Then for any integral weight $\lambda = (h_2 - 1, h_1)$ such that $h_1 < h_2$, $T_\mu^\lambda D_A$ is almost de Rham of rank 4.

Proof. In any case $T_\lambda^\mu D_A$ is a (φ, Γ)-module by Proposition 4.12 and its rank, being almost de Rham and Sen weights can be checked at points. The statements follow from the case when A is a field which was studied in [Din23, Prop. 2.19]. □

Remark 4.14. In the case when $\lambda - \mu = (1,0)$ and D_A has non-regular Sen weights, $T_\mu^\lambda D_A = D_A \otimes_L V_1$.

We start calculations of translations. The easiest case is a twist by an algebraic character.

Lemma 4.15. *For $i\in\mathbb{Z}$, let Lt^i be the algebraic character $\det^i$ of GL_2 which is also a $\mathfrak{g}$-module. Then for a rank 2 (φ,Γ)-module D_A over $\mathcal{R}_A$, t^iD_A with the standard $\mathfrak{g}$-module structure is equal to $D_A\otimes_L Lt^i$.*

Proof. On $\det^i=Lt^i$ we have $u^{\pm}.t^i=0$ and $a^{\pm}.t^i=it^i$. Thus for $x\otimes t^i, x\in D_A$ we get $u^+.(x\otimes t^i)=tx\otimes t^i, a^{\pm}.(x\otimes t^i)=(a^{\pm}+i)x\otimes t^i$ and $u^-.(x\otimes t^i)=u^-.x\otimes t^i=-\frac{P_{\mathrm{Sen}}(\nabla)}{t}x\otimes t^i$. For $t^ix\in t^iD_A$ we have $u^+.(t^ix)=t^{i+1}x, a^+.(t^ix)=t^i(a^++i)x, \mathfrak{z}.t^ix=(\gamma_1-1+2i)x$ in the notation of Lemma 4.3. Hence $a^-.t^ix=t^i(\gamma_1-1+2i-a^+-i)x=t^i((a^-+i)x)$. And $u^-.t^ix=-\frac{P'_{\mathrm{Sen}}(\nabla)}{t}t^ix=-t^i\frac{P_{\mathrm{Sen}}(\nabla)}{t}x$ where P'_{Sen} denotes the Sen polynomial of t^iD_A. The last equality comes from $\nabla.t^ix=t^i(\nabla+i)x$ and $P'_{\mathrm{Sen}}(T)=P_{\mathrm{Sen}}(T-i)=T^2-(\gamma_1+2i)T+\gamma_0+i\gamma_1+i^2$. □

By a twist, we only need to discuss the case when at least one of the weights of an almost de Rham (φ,Γ)-module is zero.

Lemma 4.16. *Let D_A be an almost de Rham (φ,Γ)-module over $\mathcal{R}_A$ with pointwisely regular Sen weights $h_1<h_2$. Then there exists $\alpha,\beta\in A$ such that $P_{\mathrm{Sen}}(T)=(T-\alpha)(T-\beta)$ and $\alpha-h_1,\beta-h_2$ are in the nilradical I of A.*

Proof. We have $P_{\mathrm{Sen}}(T)\equiv(T-h_1)(T-h_2)\mod I$. Hence $P_{\mathrm{Sen}}(h_1)\in I$. Moreover $P^{(1)}_{\mathrm{Sen}}(h_1)\equiv h_1-h_2\notin I$. By Hensel's lemma, we can find $\alpha\in A$ such that $\alpha\equiv h_1\mod I$ and $P_{\mathrm{Sen}}(\alpha)=0$. Then P_{Sen} must be of the form $(T-\alpha)(T-\beta)$ such that $\beta\equiv h_2\mod I$. □

Proposition 4.17. *Let D_A be an almost de Rham (φ,Γ)-module pointwisely with regular Sens weight $(0,k)$ and let $\lambda=(k-1,0)$ where $k\geq 1$. We assume that the Sen polynomial is equal to $(T-(k+z-h))(T-(z+h))$ for z,h in the nilradical of A (which is possible by Lemma 4.16).*

1. *Let $\mu=(k-1,k)$. The natural map $T^{\mu}_{\lambda}D_A\hookrightarrow D_A\otimes_L V_k\twoheadrightarrow D_A$ identifies $T^{\mu}_{\lambda}D_A$ with the unique almost de Rham (φ,Γ)-submodule inside D_A of weights (k,k) such that $T^{\mu}_{\lambda}D_A[\frac{1}{t}]=D_A[\frac{1}{t}]$, which contains t^kD_A and is the preimage of $(D_A/t^k)[\prod_{i=0}^{k-1}(\nabla-(k+z-h)-i)=0]$.*
2. *Let $\lambda'=(k'-1,0)$ for $k'\geq k$. The natural map $T^{\lambda'}_{\lambda}D_A\hookrightarrow D_A\otimes_L V_l\twoheadrightarrow D_A$ for $l=k'-k$ identifies $T^{\mu}_{\lambda}D_A$ with the unique almost de Rham (φ,Γ)-submodule inside D_A such that $T^{\lambda'}_{\lambda}D_A[\frac{1}{t}]=D_A[\frac{1}{t}]$ of weights $(0,k')$ and consists of $v\in D_A$ such that $\prod_{i=0}^{j}(\nabla-(z+h)-i)v\in t^{j+1}D_A$ for all $j\leq l-1$.*

In both cases, the output of the translation is a rank two (φ,Γ)-module and admits an infinitesimal character.

Proof. The last assertion follows from Corollary 4.13, or by the case when $l=1$ using the calculation below and by an induction as for [Din23, Prop. 2.19].

Let $l\geq 1$. The element $\mathfrak{z}$ acts on $D_A\otimes_L V_l$ by $2z+k+l-1$. Let e be the lowest weight vector of $\mathcal{R}^+_L/X^{l+1}\simeq V_l:\alpha\mapsto\alpha e$. Then $\mathfrak{h}=2a^+-\mathfrak{z}$ acts by $2\nabla-2z-k+1$ on D_A and by $2i-l$ on t^ie; u^+ by t on D_A and $t=\log(1+X)=\sum_{i\geq 0}\frac{(-1)^i}{i}X^i$ on V_l; u^- by $-\frac{1}{t}P_{\mathrm{Sen}}(\nabla)$ on D_A and $u^-.t^ie=i(l-i+1)t^{i-1}e$ so that $\mathfrak{c}=\mathfrak{h}^2-2\mathfrak{h}+4u^+u^-$ acts on V_l by l^2+2l. Hence the Casimir acts by (for $v\in D_A, 0\leq i\leq l$ and with the convention that $t^{i-1}e=0$ if $i=0$)

(4.1)
$$\begin{aligned}\mathfrak{c}.(v\otimes t^ie)=&((2\nabla+1-2z-k+2i-l)^2-2(2\nabla+1-2z-k+2i-l))v\otimes t^ie\\&+4i(l-i+1)(v\otimes t^ie+tv\otimes t^{i-1}e)-4P_{\mathrm{Sen}}(\nabla)v\otimes t^ie-4t^{-1}P_{\mathrm{Sen}}(\nabla)v\otimes t^{i+1}e\\=&(4(2i-l)\nabla+(2z+k-2i+l)^2-1-4(z+h)(k+z-h)+4i(l-i+1))v\otimes t^ie\\&+4i(l-i+1)tv\otimes t^{i-1}e-4t^{-1}P_{\mathrm{Sen}}(\nabla)v\otimes t^{i+1}e\\=&(4(2i-l)\nabla+(\gamma_1-2i+l)^2-1-4\gamma_0+4i(l-i+1))v\otimes t^ie\\&+4i(l-i+1)tv\otimes t^{i-1}e-4t^{-1}P_{\mathrm{Sen}}(\nabla)v\otimes t^{i+1}e\end{aligned}$$

where $\gamma_1=2z+k$ and $\gamma_0=(z+h)(k+z-h)$.

(1) Now $l=k$.

$$\begin{aligned}\mathfrak{c}.(v\otimes t^ie)=&(4(2i-k)\nabla+4(i(1-2z-k)+kz+k^2-hk+h^2)-1)v\otimes t^ie\\&+4i(k-i+1)tv\otimes t^{i-1}e-4t^{-1}P_{\mathrm{Sen}}(\nabla)v\otimes t^{i+1}e.\end{aligned}$$

Suppose that $\sum_{i=0}^{k} v_i \otimes t^i e$ is an eigenvalue for $4h^2-1$. Consider the coefficient in D_A of $1\otimes e$ we get $4ktv_1+(-4k\nabla+4(kz+k^2-hk+h^2)-1)v_0=(4h^2-1)v_0$. We see $v_1=t^{-1}(\nabla-(k+z-h))v_0$. We prove by induction that $v_i=\frac{1}{it}(\nabla-(k+z-h))v_{i-1}$ for $i\geq 1$. Suppose the statement holds for $i-1\geq 0$. Then $t^{-1}P_{\mathrm{Sen}}(\nabla)v_{i-1}=i(\nabla-(z+h)+1)v_i$. Consider the coefficient of $1\otimes t^i e$ we get

$$-4i(\nabla-(z+h)+1)v_i+(4(2i-k)\nabla+4(i(1-2z-k)+kz+k^2-hk+h^2)-1)v_i+4(i+1)(k-i)tv_{i+1}=(4h^2-1)v_i$$

which is equivalent to that $v_{i+1}=\frac{1}{(i+1)t}(\nabla-(k+z-h))v_i$ unless $i\geq k$. And if $i=k$ the equality holds under the induction assumption.

Thus

$$\prod_{i=0}^{k-1}\frac{1}{t}(\nabla-(k+z-h))v_0=\frac{1}{t^k}(\nabla-(k+z-h)-(k-1))\cdots(\nabla-(k+z-h))v_0\in D_A.$$

By the discussion in §3.3, $D'_A=\{v\in D_A\mid \frac{1}{t}(\nabla-(k+z-h))v\in D_A\}$ defines the unique sub-(φ,Γ)-module of D_A such that $D'_A[\frac{1}{t}]=D_A[\frac{1}{t}]$ and D'_A is almost de Rham of weight $(1,k)$. By Proposition A.3, the description of $T_\lambda^\mu D_A$ follows from the following statement:

Lemma 4.18. *Let M be a continuous Γ-representation over $(A\otimes_{\mathbb{Q}_p}K_m)[[t]]$ with the connection ∇ as in Appendix A such that the characteristic polynomial of ∇ on M/tM is $(X-(k+z-h))(X-(z+h))$ for $k\in\mathbb{Z}_{\geq 0}$ and h,z nilpotent. Then there is a direct sum decomposition*

$$M/t^kM=\oplus_{i=0}^{k-1}M[\nabla=(k+z-h+i)]\oplus\oplus_{i=0}^{k-1}M[\nabla=(z+h+i)]$$

where all the direct summands are projective of rank one over $A\otimes_{\mathbb{Q}_p}K_m$.

(2) Suppose that $\sum_{i=0}^{l} v_i\otimes t^i e$ is an eigenvector of $\mathfrak{c}$ for the eigenvalue $(k+l-2h)^2-1$. Consider the coefficient of $1\otimes e$ we get $4ltv_1+(-4l\nabla+(2z+k+l)^2-4(k+z-h)(z+h)-1)v_0=((k+l-2h)^2-1)v_0$. We see $v_1=\frac{1}{t}(\nabla-(h+z))v_0$. We prove by induction that we must have $v_i=\frac{1}{it}(\nabla-(h+z))v_{i-1}$ for $i\geq 1$. Assume that the statement holds for $i-1\geq 0$. Then $t^{-1}P_{\mathrm{Sen}}(\nabla)v_{i-1}=i(\nabla-(k+z-h)+1)v_i$. Consider the coefficient of $1\otimes t^ie$ we get

$$-4i(\nabla-(k+z-h)+1)v_i+(4(2i-l)\nabla+(2z+k-2i+l)^2-1-4(z+h)(k+z-h)+4i(l-i+1))v_i$$
$$+4(i+1)(l-i)tv_{i+1}=((k+l-2h)^2-1)v_i$$

which is equivalent to that $v_{i+1}=\frac{1}{(i+1)t}(\nabla-(h+z))v_i$ unless $i\geq l$. And if $i=l$ the equality holds under the statement.

To see the uniqueness. Assume that $D'_A\subset D_A$ with Sen weights $(0,k')$ and such that $D'_A[\frac{1}{t}]=D_A[\frac{1}{t}]$. Then $D_{\mathrm{pdR}}(D'_A)=D_{\mathrm{pdR}}(D_A)$ and $\mathrm{Fil}^0D_{\mathrm{pdR}}(D'_A)\subset\mathrm{Fil}^0D_{\mathrm{pdR}}(D_A)$ by definition. Since $\mathrm{Fil}^0D_{\mathrm{pdR}}(D'_A),\mathrm{Fil}^0D_{\mathrm{pdR}}(D_A)$ are both direct summand of $D_{\mathrm{pdR}}(D_A)$ of rank one, $\mathrm{Fil}^0D_{\mathrm{pdR}}(D'_A)=\mathrm{Fil}^0D_{\mathrm{pdR}}(D'_A)$ which determines $D_{\mathrm{dif}}^{m,+}(D'_A)$ for some m and also D'_A by results in Appendix A. The description holds as for (1) and by an induction on l. □

From non-regular weights to regular weights we only treat the easiest case.

Proposition 4.19. *Let Δ_A be an almost de Rham (φ,Γ)-module pointwisely with non-regular Sen weights $(0,0)$, $\mu=(-1,0)$ and $\lambda=(0,0)$. Then the natural map $T_\mu^\lambda\Delta_A\hookrightarrow\Delta_A\otimes_L V_1$ is an isomorphism. If the Sen polynomial of Δ_A is $T^2-\gamma_1T+\gamma_0\in A[T]$, then $(\mathfrak{c}-\gamma_1^2+4\gamma_0)^2=4(\gamma_1^2-4\gamma_0)$ on $\Delta_A\otimes_L V_1$.*

Proof. We follow the notation in the proof of Proposition 4.17. By (4.1) (replacing $k+2z$ by γ_1 and $(k+z-h)(z+h)$ by γ_0), we see

$$(\mathfrak{c}-\gamma_1^2+4\gamma_0).(v\otimes t^ie)=(4(2i-1)\nabla+4i(1-\gamma_1)+2\gamma_1)v\otimes t^ie+4i(2-i)tv\otimes t^{i-1}e-4t^{-1}P_{\mathrm{Sen}}(\nabla)v\otimes t^{i+1}e.$$

For $i=0$ we have

$$(\mathfrak{c}-\gamma_1^2+4\gamma_0).(v\otimes e)=(-4\nabla+2\gamma_1)v\otimes e-4t^{-1}(\nabla^2-\gamma_1\nabla+\gamma_0)v\otimes te$$

and for $i=1$

$$(\mathfrak{c}-\gamma_1^2+4\gamma_0).(v\otimes te)=(4\nabla+4-2\gamma_1)v\otimes te+4tv\otimes e.$$

Then $(\mathfrak{c}-\gamma_1^2+4\gamma_0)^2-4(\gamma_1^2-4\gamma_0)=0$ on $\Delta_A\otimes_L V_1$ by a direct check. □

The following proposition describes the counit map of the adjunction of the translations in the case $k=1$.

Proposition 4.20. *Let $\lambda=(0,0)$ and $\mu=(-1,0)$. Let D_A be an almost de Rham (φ,Γ)-module of weight $(1,0)$ over A with Sen polynomial $(T-(z-h+1))(T-(z+h))$ where z,h are nilpotent. Let $\Delta_A\subset t^{-1}D_A$ be the sub (φ,Γ)-module of Sen weights $(0,0)$ in Proposition 4.17 which provides the isomorphism $\Delta_A=T_\lambda^\mu D_A$. Let $c=\mathfrak{c}-4h^2$ which acts on D_A by $-4h$.*

Then the composite $T_\mu^\lambda\Delta_A=T_\mu^\lambda T_\lambda^\mu D_A\to D_A\hookrightarrow\Delta_A$ induced by the counit map $T_\mu^\lambda T_\lambda^\mu D_A\to D_A$ is equal to $T_\mu^\lambda\Delta_A(=\Delta_A\otimes_L\mathcal{R}_L^+/X^2)\overset{\frac14(c-4h)}{\to}T_\mu^\lambda\Delta_A[c+4h]\subset\Delta_A\otimes_L\mathcal{R}_L^+/X^2\twoheadrightarrow\Delta_A$.

Proof. Before the proof, note that Proposition 4.17 shows that Δ_A contains D_A and identifies Δ_A with the preimage of $(t^{-1}D_A/D_A)[\nabla=z-h]$. Also by Proposition 4.19, $(c-4h)(c+4h)$ acts as zero on $T_\mu^\lambda\Delta_A$.

Recall we identify $\mathcal{R}_L^+/X^2\simeq\mathrm{Sym}^1L^2=Lx\oplus Ly$ where $1=y$ is the lowest weight vector, $t=X=x$. Let $(\mathrm{Sym}^1L^2)^\vee=Lx^*\oplus Ly^*$ be the dual where x^*,y^* are the dual basis. We fix an isomorphism $Lt^{-1}\otimes_L\mathcal{R}_L^+/X^2\simeq\det^{-1}\otimes_L\mathrm{Sym}^1L^2=(\mathrm{Sym}^1L^2)^\vee$ by $t^{-1}\otimes1=x^*,t^{-1}\otimes t=-y^*$ (since $u^+.x^*=-y^*$). The unit and counit map is induced by

$$\begin{aligned}&L\to(Lx^*\oplus Ly^*)\otimes_L(Lx\oplus Ly)\to L\\&1\mapsto x^*\otimes x+y^*\otimes y\end{aligned}$$

where the second map is the evaluation map.

The map $T_\mu^\lambda\Delta_A=T_\mu^\lambda T_\lambda^\mu D_A\to D_A\hookrightarrow\Delta_A$ factors through (by Lemma 4.15)

$$T_\mu^\lambda T_\lambda^\mu D_A=\Delta_A\otimes_L\mathcal{R}_L^+/X^2\hookrightarrow t^{-1}D_A\otimes_L\mathcal{R}_L^+/X^2\otimes_L\mathcal{R}_L^+/X^2\to D_A\hookrightarrow\Delta_A$$

where $v\in\Delta_A$ is sent to $v\otimes e+t^{-1}(\nabla-(z-h))v\otimes te$ in $t^{-1}D_A\otimes_L\mathcal{R}_L^+/X^2$ by the proof of Proposition 4.17. The image of $(v\otimes e+t^{-1}(\nabla-(z-h))v\otimes te)\otimes e\in(t^{-1}D_A\otimes_L\mathcal{R}_L^+/X^2)\otimes_L\mathcal{R}_L^+/X^2$ via the evaluation map in D_A is $-(\nabla-(z-h))v$ and the image of $(v\otimes e+t^{-1}(\nabla-(z-h))v\otimes te)\otimes te$ is tv.

On the other hand, for $v\in\Delta_A$, using (4.1) again for $\Delta_A\otimes_L\mathcal{R}_L^+/X^2$,

$$(c\pm4h).(v\otimes t^ie)=(4(2i-1)\nabla+4i(1-2z)+4z\pm4h)v\otimes t^ie+4i(2-i)tv\otimes t^{i-1}e-4t^{-1}P_{\mathrm{Sen}}(\nabla)v\otimes t^{i+1}e.$$

For $i=0$ we have

$$(c-4h).(v\otimes e)=4(-\nabla+(z-h))v\otimes e-4t^{-1}P_{\mathrm{Sen}}(\nabla)v\otimes te.$$

and

$$(c-4h).(v\otimes te)=4(\nabla+(1-z-h))v\otimes te+4tv\otimes e.$$

Hence the image of $\frac14(c-4h)(v\otimes e)$ in Δ_A is $(-\nabla+z-h)v$ and the image of $\frac14(c-4h)(v\otimes te)$ in Δ_A is tv. ☐

5. Geometric translations for $\mathrm{GL}_2(\mathbb{Q}_p)$

We will prove our main results on the geometric translations in the $\mathrm{GL}_2(\mathbb{Q}_p)$ case (Theorem 5.15). We first make necessary preparations in §5.1 for the study of the direct image of (φ,Γ)-modules over formal rigid spaces in Construction 5.4. We prove our main result in §5.2 which compares the direct image and the translation. The interested reader may consult the pointwise cases in §5.3 first or the description in Corollary 5.12 for the basic ideas on computations. We take $G=\mathrm{GL}_{2/L}$, $K=\mathbb{Q}_p$ and $f:\widetilde{\mathfrak{g}}\to\mathfrak{g}$.

5.1. **Formal completion of (φ,Γ)-modules.**

A (φ,Γ)-module D_A^r over $\mathcal{R}_A^r$ for an affinoid algebra A over L is equivalently a (φ,Γ)-bundle on $\mathbb{U}^r_{\mathrm{Sp}(A)}=\mathrm{Sp}(A)\times_L\mathbb{U}^r$, namely a Γ-equivariant vector bundle $\mathcal{D}^r_{\mathrm{Sp}(A)}$ on $\mathbb{U}^r_{\mathrm{Sp}(A)}$ equipped with an isomorphism $\varphi^*\mathcal{D}^r_{\mathrm{Sp}(A)}\simeq\mathcal{D}^r_{\mathrm{Sp}(A)}|_{\mathbb{U}^{r/p}_{\mathrm{Sp}(A)}}$ commuting with Γ-actions. This point of view will be more convenient for the consideration of cohomologies (but will not work for translations), and we need a similar description for formal completions of (φ,Γ)-modules. A basic discussion on coherent modules over formal rigid spaces can be found in Appendix B.

Let $\mathrm{Sp}(A)\in\mathrm{Rig}_L$ with an ideal I of A. Let $\mathfrak{Y}_n=\mathrm{Sp}(A/I^n)$ and $\mathfrak{Y}^\wedge:=\varinjlim_n\mathfrak{Y}_n$. The latter is a ringed site with the structure sheaf $\mathcal{O}_{\mathfrak{Y}^\wedge}=\varprojlim_n\mathcal{O}_{\mathfrak{Y}_n}$ whose global section is $A^\wedge:=\varprojlim_n A/I^n$. There is a sheaf of $\mathcal{O}_{\mathfrak{Y}^\wedge}$-module $\mathcal{R}^r_{\mathfrak{Y}^\wedge}:=\varprojlim_n\mathcal{R}^r_{\mathfrak{Y}_n}$ for small enough $r>0$ whose global section is $\mathcal{R}^r_{A^\wedge}:=\varprojlim_n\mathcal{R}^r_{A/I^n}$. We define $\mathcal{R}_{A^\wedge}:=\varinjlim_r\mathcal{R}^r_{A^\wedge}$.

Recall that for r small enough and $s < r$, $\varphi : \mathcal{R}_A^{[s,r]} \to \mathcal{R}_A^{[s/p,r/p]}, \mathcal{R}_A^r \to \mathcal{R}_A^{r/p}$ make the targets finite free of rank p over the sources and we have $\mathcal{R}_A^{r/p} = \oplus_{i=0}^{p-1}(1+X)^i\varphi(\mathcal{R}_A^r)$. Taking I-adic completion, we get $\mathcal{R}_{A^\wedge}^{r/p} = \oplus_{i=0}^{p-1}(1+X)^i\varphi(\mathcal{R}_{A^\wedge}^r)$ and taking direct limit $\mathcal{R}_{A^\wedge} = \oplus_{i=0}^{p-1}(1+X)^i\varphi(\mathcal{R}_{A^\wedge})$.

Definition 5.1. Let $\mathfrak{X}^\wedge$ be the formal completion of a rigid space $\mathfrak{X}$ along a Zariski closed subspace defined by a coherent ideal sheaf $\mathcal{I}$ as in Definition B.1.

(1) A (φ,Γ)-module (resp. $(\varphi,\Gamma,\mathfrak{g})$-module) $D_{\mathfrak{X}^\wedge}^r$ over $\mathcal{R}_{\mathfrak{X}^\wedge}^r := \varprojlim_n \mathcal{R}_{\mathfrak{X}_n}^r$ is a locally finite projective $\mathcal{R}_{\mathfrak{X}^\wedge}^r$-module with a (φ,Γ)-structure, i.e., an isomorphism
$$\varphi : \varphi^* D_{\mathfrak{X}^\wedge}^r := \mathcal{R}_{\mathfrak{X}^\wedge}^{r/p} \otimes_{\varphi,\mathcal{R}_{\mathfrak{X}^\wedge}^r} D_{\mathfrak{X}^\wedge}^r \simeq \mathcal{R}_{\mathfrak{X}^\wedge}^{r/p} \otimes_{\mathcal{R}_{\mathfrak{X}^\wedge}^r} D_{\mathfrak{X}^\wedge}^r$$
commuting with a semilinear action of Γ (resp. (φ,Γ)-structure and an action of $\mathfrak{g}$) such that there exist (φ,Γ)-modules (resp. $(\varphi,\Gamma,\mathfrak{g})$-modules as in Definition 4.2) $D_{\mathfrak{X}_n}^r$ over $\mathcal{R}_{\mathfrak{X}_n}^r$ satisfying $D_{\mathfrak{X}_{n+1}}^r/\mathcal{I}^n = D_{\mathfrak{X}_n}^r$ for all $n \geq 1$ and there exists an isomorphism $D_{\mathfrak{X}^\wedge}^r \simeq \varprojlim_n D_{\mathfrak{X}_n}^r$.

(2) A (φ,Γ)-module $D_{\mathfrak{X}^\wedge}$ over $\mathcal{R}_{\mathfrak{X}^\wedge} := \varinjlim_r \mathcal{R}_{\mathfrak{X}^\wedge}^r$ is a $\mathcal{R}_{\mathfrak{X}^\wedge}$-module with (φ,Γ)-structure such that there exists a (φ,Γ)-module $D_{\mathfrak{X}^\wedge}^r$ over $\mathcal{R}_{\mathfrak{X}^\wedge}^r$ and $D_{\mathfrak{X}^\wedge} = \mathcal{R}_{\mathfrak{X}^\wedge} \otimes_{\mathcal{R}_{\mathfrak{X}^\wedge}^r} D_{\mathfrak{X}^\wedge}^r = \varinjlim_{r'<r} \mathcal{R}_{\mathfrak{X}^\wedge}^{r'} \otimes_{\mathcal{R}_{\mathfrak{X}^\wedge}^r} D_{\mathfrak{X}^\wedge}^r$.

The underlying $\mathcal{R}_{\mathfrak{Y}}^r$-module of a (φ,Γ)-module $D_{\mathfrak{Y}}^r$ over $\mathcal{R}_{\mathfrak{Y}}^r$ is the $\mathcal{R}_{\mathfrak{Y}}^r$-module associated to its global section D_A^r which is finite projective over $\mathcal{R}_A^r$: for any affinoid open $\mathrm{Sp}(B) \subset \mathfrak{Y}$, $\Gamma(\mathrm{Sp}(B), D_{\mathfrak{Y}}^r) = D_A^r \otimes_{\mathcal{R}_A^r} \mathcal{R}_B^r$. A vector bundle over $\mathbb{U}_{\mathfrak{Y}}^r$ is equivalently a compatible family of finite projective modules over $\mathcal{R}_{\mathfrak{Y}}^{[s,r]}$ or $\mathcal{R}_A^{[s,r]}$ for $s < r$. We can similarly define the notion of vector bundles or φ-bundles over $\mathbb{U}_{\mathfrak{Y}^\wedge}^r = \varinjlim_{s,n} \mathbb{U}_{\mathfrak{Y}_n}^{[s,r]}$.

Lemma 5.2. *Let $g : \mathbb{U}_{\mathfrak{Y}^\wedge}^r \to \mathfrak{Y}^\wedge$ be the projection.*

(1) Suppose that $D_{A^\wedge}^r$ is a finite projective $\mathcal{R}_{A^\wedge}^r$-module, then $D_{A^\wedge}^r$ is I-adically complete and is the global section of a vector bundle $(\mathcal{R}_{A/I^n}^{[s,r]} \otimes_{\mathcal{R}_{A^\wedge}^r} D_{A^\wedge}^r)_{s,n}$ over $\mathbb{U}_{\mathfrak{Y}^\wedge}^r$. Moreover for $r' \leq r$, $\mathcal{R}_{A^\wedge}^{r'} \otimes_{\mathcal{R}_{A^\wedge}^r} D_{A^\wedge}^r = \varprojlim_n D_{A/I_n}^{r'}$.

(2) Taking global sections on $\mathfrak{Y}^\wedge$ induces an equivalence between the category of $\mathcal{R}_{\mathfrak{Y}^\wedge}^r$-modules of the form $D_{\mathfrak{Y}^\wedge}^r = \varprojlim_n D_{\mathfrak{Y}_n}^r$ where $D_{\mathfrak{Y}_n}^r$ are $\mathcal{R}_{\mathfrak{Y}_n}^r$-modules associated to projective $\mathcal{R}_{A/I^n}^r$-modules and $D_{\mathfrak{Y}_{n+1}}^r/I^n = D_{\mathfrak{Y}_n}^r$ for all n and the category of finite projective $\mathcal{R}_{A^\wedge}^r$-modules.

(3) The direct image functor g_* induces an equivalence of categories between φ-bundles over $\mathbb{U}_{\mathfrak{Y}^\wedge}^r$ and finite projective φ-modules over $\mathcal{R}_{\mathfrak{Y}^\wedge}^r$.

Proof. (1) If $D_{A^\wedge}^r$ is a finite projective $\mathcal{R}_{A^\wedge}^r$-module, we may find another finite module D' such that $D_{A^\wedge}^r \oplus D' \simeq (\mathcal{R}_{A^\wedge}^r)^n$ for some n. Since $(\mathcal{R}_{A^\wedge}^r)^n$ is I-adically complete [Sta24, Tag 05GG], so is its direct summand. Hence $D_{A^\wedge}^r = \varprojlim_n D_{A/I^n}^r = \varprojlim_{n,s} D_{A/I^n}^{[s,r]} = \varprojlim_{n,s} D_{A/I^n}^r \otimes_{\mathcal{R}_{A/I^n}^r} \mathcal{R}_{A/I^n}^{[s,r]} = \varprojlim_s (D_{A^\wedge}^r \otimes_{\mathcal{R}_{A^\wedge}^r} \mathcal{R}_{A^\wedge}^{[s,r]})$ (for the last equality we use that $D_{A^\wedge}^r \otimes_{\mathcal{R}_{A^\wedge}^r} \mathcal{R}_{A^\wedge}^{[s,r]}$ is I-adically complete being finite projective over $\mathcal{R}_{A^\wedge}^{[s,r]}$). The statement for $D_{A^\wedge}^{r'}$ follows similarly.

(2) Suppose that $(D_{\mathfrak{Y}_n}^r)_n$ is a collection of finite projective $\mathcal{R}_{\mathfrak{Y}_n}^r$-modules (giving vector bundles on $\mathbb{U}_{\mathfrak{Y}_n}^r$) such that $D_{\mathfrak{Y}^\wedge}^r := \varprojlim_n D_{\mathfrak{Y}_n}^r$. Write $D_{A/I^n}^r := \Gamma(\mathfrak{Y}_n, D_{\mathfrak{Y}_n}^r)$. By the definition of inverse limit of sheaves, for any affinoid open $\mathrm{Sp}(B) \subset \mathfrak{Y}$, $\Gamma(\mathrm{Sp}(B/I), D_{\mathfrak{Y}^\wedge}^r) = \varprojlim_n \Gamma(\mathrm{Sp}(B/I^n), D_{\mathfrak{Y}_n}^r) = D_{B^\wedge}^r := \varprojlim_n \mathcal{R}_{B/I^n}^r \otimes_{\mathcal{R}_{A/I^n}^r} D_{A/I^n}^r$. Hence the section is finite projective over $\mathcal{R}_{B^\wedge}^r$ and $D_{B^\wedge}^r/I^n = \mathcal{R}_{B/I^n}^r \otimes_{\mathcal{R}_{A/I^n}^r} D_{A/I^n}^r$ by (2) of Lemma B.7. Then $D_{B^\wedge}^r = D_{A^\wedge}^r \otimes_{\mathcal{R}_{A^\wedge}^r} \mathcal{R}_{B^\wedge}^r$ (both sides are I-adically complete by (1)). Take $B = A$ we see the global section is finite projective. The essential surjectivity follows from (1): given a (φ,Γ)-module $D_{A^\wedge}^r$ finite projective over $\mathcal{R}_{A^\wedge}^r$, $D_{A^\wedge}^r/I^n$ defines sheaves $D_{\mathfrak{Y}_n}^r$ and $\Gamma(\mathfrak{Y}^\wedge, \varprojlim_n D_{\mathfrak{Y}_n}^r) = D_{A^\wedge}^r$. The fully faithfulness follows as for (2) of Lemma B.4.

(3) Let $\mathcal{D}_{\mathfrak{Y}^\wedge}^r = (\mathcal{D}_{\mathfrak{Y}_n}^{[s,r]})_{s,n}$ be a φ-bundle on $\mathbb{U}_{\mathfrak{Y}^\wedge}^r$. By [KPX14, Prop. 2.2.7], the global section of $(\mathcal{D}_{\mathfrak{Y}_n}^{[s,r]})_s$ for a fix n defines via g_* a finite projective φ-module denoted by $D_{\mathfrak{Y}_n}^r$ over $\mathcal{R}_{\mathfrak{Y}_n}^r$. By the equivalence of *loc. cit.*, we have $D_{\mathfrak{Y}_n}^r/I^{n-1} = D_{\mathfrak{Y}_{n-1}}^r$. Hence the $\mathcal{R}_{\mathfrak{Y}^\wedge}^r$-module $D_{\mathfrak{Y}^\wedge}^r := \varprojlim_n D_{\mathfrak{Y}_n}^r$ is finite projective over $\mathcal{R}_{\mathfrak{Y}^\wedge}^r$ by (2). Thus g_* sends φ-bundles to fintie projective φ-modules (see also Lemma 5.8 below). The fully faithfulness of g_* follows from mod I^n-cases in [KPX14, Prop.

2.2.7] and that for two φ-modules D_1, D_2 (which are I-adically complete by (1))

$$\mathrm{Hom}_{\mathcal{R}^r_{\mathfrak{Y}^\wedge}}(D_1^r, D_2^r) = \varprojlim_n \mathrm{Hom}_{\mathcal{R}^r_{\mathfrak{Y}^\wedge}}(D_1^r, D_2^r/I^n) = \varprojlim_n \mathrm{Hom}_{\mathcal{R}^r_{\mathfrak{Y}_n}}(D_1^r/I^n, D_2^r/I^n).$$

The essential surjectivity follows from (1) and (2). □

Remark 5.3. Since we will only deal with finite projective modules, we ignore if we can define the notion of coadmissible modules over the topological ring $\mathcal{R}^r_{A^\wedge} = \varprojlim_{s,n} \mathcal{R}^{[s,r]}_{A/I^n}$ as for modules over $\mathcal{R}^r_A$ in [ST03, §3].

We construct below our major players: some (φ,Γ)-module $D^r_{\widetilde{\mathfrak{Y}}^\wedge}$ over a formal rigid space $\widetilde{\mathfrak{Y}}^\wedge$ projective over $\mathfrak{Y}^\wedge$ and we will study its direct image along the map $\widetilde{\mathfrak{Y}}^\wedge \to \mathfrak{Y}^\wedge$.

Construction 5.4. Fix $\mathbf{h} = (h_1, h_2) \in \mathbb{Z}^2, h_1 < h_2$. Let $\mathfrak{Y} = \mathrm{Sp}(A) \in \mathrm{Rig}_L$ be an affinoid with a (φ,Γ)-module Δ_A of rank 2 over $\mathfrak{Y}$ base changed from a (φ,Γ)-module Δ^r_A over $\mathcal{R}^r_A$ for some $r > 0$. We assume that r is taken such that the number $m(r)$ is large enough for $D^{m(r),+}_{\mathrm{dif}}(\Delta^r_A)$ in the sense of Definition A.4.

Let I be the ideal of A generated by zeroth and the first coefficients of the Sen polynomial of Δ_A. Let $\mathfrak{Y}_n = \mathrm{Sp}(A/I^n)$ and $\mathfrak{Y}^\wedge = \varinjlim \mathfrak{Y}_n$. By definition, $\mathfrak{Y}^\wedge = \mathfrak{Y} \times_{\mathfrak{X}_2} (\mathfrak{X}_2)^\wedge_0$. According to Proposition 3.12, $\widetilde{\mathfrak{Y}}^\wedge := f_{\mathbf{h}}^{-1}(\mathfrak{Y}^\wedge) = \mathfrak{Y}^\wedge \times_{(\mathfrak{X}_2)^\wedge_0} (\mathfrak{X}_2)^\wedge_{\mathbf{h}} = \mathfrak{Y}^\wedge \times_{\mathfrak{g}/G} \widetilde{\mathfrak{g}}/G$. In the following, we construct explicitly these spaces and the universal (φ,Γ)-module $D_{\widetilde{\mathfrak{Y}}^\wedge}$ over $\mathcal{R}_{\widetilde{\mathfrak{Y}}^\wedge}$.

$$\begin{array}{ccccc} \widetilde{\mathfrak{Y}}^\wedge & \longrightarrow & (\mathfrak{X}_2)^\wedge_{\mathbf{h}} & \xrightarrow{D_{\mathrm{pdR}}} & \widetilde{\mathfrak{g}}/G \\ \downarrow{\scriptstyle f_{\mathbf{h}}} & & \downarrow{\scriptstyle f_{\mathbf{h}}} & & \downarrow{\scriptstyle f} \\ \mathfrak{Y}^\wedge & \longrightarrow & (\mathfrak{X}_2)^\wedge_0 & \xrightarrow{D_{\mathrm{pdR}}} & \mathfrak{g}/G. \end{array}$$

Write $\Delta^r_{\mathfrak{Y}_n}$ for the base change of Δ^r_A. Proposition A.6 gives a vector bundle $D_{\mathrm{pdR}}(\Delta_{\mathfrak{Y}_n})$ on $\mathfrak{Y}_n$ together with a nilpotent operator $\nu_{\mathfrak{Y}_n}$ such that $D_{\mathrm{pdR}}(\Delta_{\mathfrak{Y}_n}) \otimes_{\mathcal{O}_{\mathfrak{Y}_n}} \mathcal{O}_{\mathfrak{Y}_{n-1}} = D_{\mathrm{pdR}}(\Delta_{\mathfrak{Y}_{n-1}})$ for $n \geq 2$. Let $\mathfrak{Y}_n^\square$ be the GL_2-torsor over $\mathfrak{Y}_n$ trivializing $D_{\mathrm{pdR}}(D_{\mathfrak{Y}_n})$. Then $\mathfrak{Y}_n^\square \times_{\mathfrak{Y}_n} \mathfrak{Y}_{n-1} = \mathfrak{Y}_{n-1}^\square$. The nilpotent operator $\nu_{\mathfrak{Y}_n}$ induces $\mathfrak{Y}_n^\square \to \mathfrak{g}$. Let $\widetilde{\mathfrak{Y}}_n^\square = \mathfrak{Y}_n^\square \times_{\mathfrak{g}} \widetilde{\mathfrak{g}}$ and $\widetilde{\mathfrak{Y}}_n = [\widetilde{\mathfrak{Y}}_n^\square/\mathrm{GL}_2]$. Then $\widetilde{\mathfrak{Y}}_n$ is a rigid analytic space projective over $\mathfrak{Y}_n$ and $\widetilde{\mathfrak{Y}}_n \times_{\mathfrak{Y}_n} \mathfrak{Y}_{n-1} = \widetilde{\mathfrak{Y}}_{n-1}$. Let $\widetilde{\mathfrak{Y}}^\wedge = \varinjlim \widetilde{\mathfrak{Y}}_n^\wedge$ (see Remark 5.5 below). Let $\Delta^r_{\widetilde{\mathfrak{Y}}_n}$ be the pullback and $\Delta^r_{\widetilde{\mathfrak{Y}}^\wedge} := \varprojlim_n \Delta^r_{\widetilde{\mathfrak{Y}}_n}$.

On each $\widetilde{\mathfrak{Y}}_n$, the universal ν-stable filtration of $D_{\mathrm{pdR}}(\Delta_{\widetilde{\mathfrak{Y}}_n})$ provided from $\widetilde{\mathfrak{g}}$ gives a Γ-invariant $(K_m \otimes_{\mathbb{Q}_p} \mathcal{O}_{\widetilde{\mathfrak{Y}}_n})[[t]]$-lattice of weight $\mathbf{h}$ inside $D^m_{\mathrm{dif}}(\Delta_{\widetilde{\mathfrak{Y}}_n})$ by Proposition A.6, and by Proposition A.3 we obtain a modification $D^r_{\widetilde{\mathfrak{Y}}_n}$ of $\Delta^r_{\widetilde{\mathfrak{Y}}_n}$ on $\widetilde{\mathfrak{Y}}_n$ which is a (φ,Γ)-module over $\mathcal{R}^r_{\widetilde{\mathfrak{Y}}_n}$. Let $D^r_{\widetilde{\mathfrak{Y}}^\wedge} = \varprojlim_n D^r_{\widetilde{\mathfrak{Y}}_n}$. We write $\mathcal{D}^r_{\widetilde{\mathfrak{Y}}_n^\wedge}$ for $D^r_{\widetilde{\mathfrak{Y}}_n^\wedge}$ viewed as coherent sheaves on $\mathbb{U}^r_{\widetilde{\mathfrak{Y}}_n}$ and similarly $\mathcal{D}^r_{\widetilde{\mathfrak{Y}}^\wedge} = \varprojlim_n \mathcal{D}^r_{\widetilde{\mathfrak{Y}}_n}$.

Finally, let $D_{\widetilde{\mathfrak{Y}}^\wedge} = \varinjlim_r D^r_{\widetilde{\mathfrak{Y}}^\wedge}$, a (φ,Γ)-module in the sense of Definition 5.1.

Remark 5.5. With the trivialization of $D_{\mathrm{pdR}}(D_{A^\wedge})$ as in Remark 3.29, ν induces maps $\mathfrak{Y}_n \to \mathfrak{g}$ and $\widetilde{\mathfrak{Y}}_n = \mathfrak{Y}_n \times_{\mathfrak{g}} \widetilde{\mathfrak{g}}$. And $\widetilde{\mathfrak{Y}}^\wedge = \varinjlim \mathfrak{Y}_n \times_{\mathfrak{g}} \widetilde{\mathfrak{g}}$. We also note that in this (local) case, $A^\wedge$-linear operator on $(A^\wedge)^2 = D_{\mathrm{pdR}}(D_{A^\wedge})$ induces a scheme map $\mathrm{Spec}(A^\wedge) \to \mathfrak{g}^{\mathrm{alg}}$. The proper formal rigid space $\widetilde{\mathfrak{Y}}^\wedge$ over $\mathfrak{Y}^\wedge$ is the relative analytification of the formal completion of a projective scheme $\mathrm{Spec}(A^\wedge) \times_{\mathfrak{g}^{\mathrm{alg}}} \widetilde{\mathfrak{g}}^{\mathrm{alg}} \subset \mathbf{P}^1_{\mathrm{Spec}(A^\wedge)}$ over $\mathrm{Spec}(A^\wedge)$ in the way discussed before Corollary B.6.

By discussions in Appendix B, $\mathcal{D}^r_{\widetilde{\mathfrak{Y}}^\wedge} = \varprojlim_n \mathcal{D}^r_{\widetilde{\mathfrak{Y}}_n}$ is a coherent $\mathcal{O}_{\mathbb{U}^r_{\widetilde{\mathfrak{Y}}^\wedge}}$-module and is locally free of rank two (Lemma 5.2).

We still write $f_{\mathbf{h}} : \widetilde{\mathfrak{Y}}^\wedge \to \mathfrak{Y}^\wedge$ for the morphism of ringed spaces with the sheaves of abstract rings $\mathcal{R}^r_{\widetilde{\mathfrak{Y}}^\wedge}$ or $\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge}$, and $\mathcal{R}^r_{\mathfrak{Y}^\wedge}$ or $\mathcal{O}_{\mathfrak{Y}^\wedge}$, via the map $f_{\mathbf{h}}^{-1}\mathcal{R}^r_{\mathfrak{Y}^\wedge} \to \mathcal{R}^r_{\widetilde{\mathfrak{Y}}^\wedge}$ or $f_{\mathbf{h}}^{-1}\mathcal{O}_{\mathfrak{Y}^\wedge} \to \mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge}$.

Definition 5.6. We write $Rf_{\mathbf{h},n,*}D^r_{\widetilde{\mathfrak{Y}}_n}$, resp. $Rf_{\mathbf{h},*}D^r_{\widetilde{\mathfrak{Y}}^\wedge}$, resp. $Rf_{\mathbf{h},*}D_{\widetilde{\mathfrak{Y}}^\wedge}$ to be the derived direct image of the sheaf of $\mathcal{O}_{\widetilde{\mathfrak{Y}}_n}$-module $D^r_{\widetilde{\mathfrak{Y}}_n}$, resp. $\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge}$-module $D^r_{\widetilde{\mathfrak{Y}}^\wedge}$, resp. $\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge}$-module $D_{\widetilde{\mathfrak{Y}}^\wedge}$ along the map $f_{\mathbf{h}}$ [Sta24, Tag 071J].

Remark 5.7. The direct image will only work well after Lemma 5.8 below. The sheaves $f_{\mathbf{h},n,*}D^r_{\widetilde{\mathfrak{Y}}_n}$ and $f_{\mathbf{h},*}D^r_{\widetilde{\mathfrak{Y}}^\wedge}$ remain to have the Γ-actions but may not be (φ,Γ)-modules as in Definition 5.1, namely they may not be projective over $\mathcal{R}^r_{\mathfrak{Y}_n}$ or $\mathcal{R}^r_{\mathfrak{Y}^\wedge}$ and the map

$$\varphi : R^i f_{\mathbf{h},*} D^r_{\widetilde{\mathfrak{Y}}^\wedge} \to R^i f_{\mathbf{h},*}(\mathcal{R}^{r/p}_{\widetilde{\mathfrak{Y}}^\wedge} \otimes_{\mathcal{R}^r_{\widetilde{\mathfrak{Y}}^\wedge}} D^r_{\widetilde{\mathfrak{Y}}^\wedge})$$

induced by $\varphi : D^r_{\widetilde{\mathfrak{Y}}^\wedge} \to \mathcal{R}^{r/p}_{\widetilde{\mathfrak{Y}}^\wedge} \otimes_{\mathcal{R}^r_{\widetilde{\mathfrak{Y}}^\wedge}} D^r_{\widetilde{\mathfrak{Y}}^\wedge}$ a priori may not factor through $\mathcal{R}^{r/p}_{\mathfrak{Y}^\wedge} \otimes_{\mathcal{R}^r_{\mathfrak{Y}^\wedge}} R^i f_{\mathbf{h},*} D^r_{\widetilde{\mathfrak{Y}}^\wedge}$.

We use $f^r_{\mathbf{h},n} : \mathbb{U}^r_{\widetilde{\mathfrak{Y}}_n} \to \mathbb{U}^r_{\mathfrak{Y}_n}$ to denote the base change of $f_{\mathbf{h}}$ and similarly $f^r_{\mathbf{h}} : \mathbb{U}^r_{\widetilde{\mathfrak{Y}}^\wedge} \to \mathbb{U}^r_{\mathfrak{Y}^\wedge}$. By Corollary B.6, $Rf^r_{\mathbf{h},*}$ sends coherent $\mathcal{O}_{\mathbb{U}^r_{\widetilde{\mathfrak{Y}}^\wedge}}$-modules to coherent $\mathcal{O}_{\mathbb{U}^r_{\mathfrak{Y}^\wedge}}$-modules. Consider the diagram

$$\begin{array}{ccc} \mathbb{U}^r_{\widetilde{\mathfrak{Y}}^\wedge} & \xrightarrow{\widetilde{g}} & \widetilde{\mathfrak{Y}}^\wedge \\ \downarrow{\scriptstyle f^r_{\mathbf{h}}} & & \downarrow{\scriptstyle f_{\mathbf{h}}} \\ \mathbb{U}^r_{\mathfrak{Y}^\wedge} & \xrightarrow{g} & \mathfrak{Y}^\wedge \end{array}$$

of ringed sites with structure sheaves $\mathcal{O}_{\mathbb{U}^r_{\widetilde{\mathfrak{Y}}^\wedge}}, \mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge}$, etc.

Lemma 5.8. *Let $D_{\widetilde{\mathfrak{Y}}^\wedge}, \mathcal{D}_{\widetilde{\mathfrak{Y}}^\wedge}$ be as in Construction 5.4.*

(1) For each $n \geq 1$, $R\widetilde{g}_* \mathcal{D}^r_{\widetilde{\mathfrak{Y}}_n} = D^r_{\widetilde{\mathfrak{Y}}_n}$ as modules over $\mathcal{R}^r_{\widetilde{\mathfrak{Y}}_n}$. Similarly $R\widetilde{g}_*\mathcal{D}^r_{\widetilde{\mathfrak{Y}}^\wedge} = D^r_{\widetilde{\mathfrak{Y}}^\wedge}$. Hence $Rf_{\mathbf{h},*}D^r_{\widetilde{\mathfrak{Y}}^\wedge} = Rg_* Rf^r_{\mathbf{h},*}\mathcal{D}^r_{\widetilde{\mathfrak{Y}}^\wedge}$ and $Rf_{\mathbf{h},*}D^r_{\widetilde{\mathfrak{Y}}_n} = Rg_*Rf^r_{\mathbf{h},*}\mathcal{D}^r_{\widetilde{\mathfrak{Y}}_n}$ for all $n \geq 1$.

(2) As sheaves of $\mathcal{R}^r_{\mathfrak{Y}^\wedge}$-modules with Γ-actions, $R^i f_{\mathbf{h},*} D^r_{\widetilde{\mathfrak{Y}}^\wedge} = g_* R^i f^r_{\mathbf{h},*}\mathcal{D}^r_{\widetilde{\mathfrak{Y}}^\wedge}$ and are isomorphic to the inverse limit $\varprojlim_n g_* R^i f^r_{\mathbf{h},*}\mathcal{D}^r_{\widetilde{\mathfrak{Y}}_n}$.

(3) For $r' \leq r$ and $i \geq 0$, the natural map $\mathcal{R}^{r'}_{\mathfrak{Y}^\wedge} \otimes_{\mathcal{R}^r_{\mathfrak{Y}^\wedge}} R^i f_{\mathbf{h},*} D^r_{\widetilde{\mathfrak{Y}}^\wedge} \to R^i f_{\mathbf{h},*}(\mathcal{R}^{r'}_{\widetilde{\mathfrak{Y}}^\wedge} \otimes_{\mathcal{R}^r_{\widetilde{\mathfrak{Y}}^\wedge}} D^r_{\widetilde{\mathfrak{Y}}^\wedge})$ is an isomorphism provided that $R^i f^r_{\mathbf{h},*}\mathcal{D}^r_{\widetilde{\mathfrak{Y}}^\wedge}$ is a locally finite free $\mathcal{O}_{\mathbb{U}^r_{\mathfrak{Y}^\wedge}}$-module. In this case, $R^i f_{\mathbf{h},*} D^r_{\widetilde{\mathfrak{Y}}^\wedge}$ is a (φ, Γ)-module over $\mathcal{R}^r_{\mathfrak{Y}^\wedge}$.

(4) For each i, $R^i f_{\mathbf{h},*} D_{\widetilde{\mathfrak{Y}}^\wedge} = \varinjlim_r R^i f_{\mathbf{h},*} D^r_{\widetilde{\mathfrak{Y}}^\wedge}$. Under the assumption in (3), the direct image $R^i f_{\mathbf{h},*} D_{\widetilde{\mathfrak{Y}}^\wedge}$ is a (φ, Γ)-module and $R^i f_{\mathbf{h},*} D_{\widetilde{\mathfrak{Y}}^\wedge} = \mathcal{R}_{\mathfrak{Y}^\wedge} \otimes_{\mathcal{R}^r_{\mathfrak{Y}^\wedge}} R^i f_{\mathbf{h},*} D^r_{\widetilde{\mathfrak{Y}}^\wedge}$.

Proof. (1) The first statement is classical and follows from the vanishing of higher coherent cohomologies of quasi-Stein spaces $\mathbb{U}^r$ [Kie67]. The derived direct image commutes with derived inverse limit [Sta24, Tag 0BKP]. The inverse systems $(\mathcal{D}^r_{\widetilde{\mathfrak{Y}}_n})_n$ and $(D^r_{\widetilde{\mathfrak{Y}}_n})_n$ are Mittag-Leffler. We get that $R\widetilde{g}_*\mathcal{D}^r_{\widetilde{\mathfrak{Y}}^\wedge} = R\widetilde{g}_* R\varprojlim_n \mathcal{D}^r_{\widetilde{\mathfrak{Y}}_n} = R\varprojlim_n R\widetilde{g}_*\mathcal{D}^r_{\widetilde{\mathfrak{Y}}_n} = D^r_{\widetilde{\mathfrak{Y}}^\wedge}$.

(2) By Corollary B.6 (and its proof), for each $i \geq 0$, $R^i f^r_{\mathbf{h},*}\mathcal{D}^r_{\widetilde{\mathfrak{Y}}^\wedge}$ is a coherent $\mathcal{O}_{\mathbb{U}^r_{\mathfrak{Y}^\wedge}}$-module and $R^i f^r_{\mathbf{h},*}\mathcal{D}^r_{\widetilde{\mathfrak{Y}}^\wedge} = \varprojlim_n R^i f^r_{\mathbf{h},*}\mathcal{D}^r_{\widetilde{\mathfrak{Y}}_n}$ is an inverse limit of a Mittag-Leffler inverse system. We see

$$\begin{aligned} Rg_* R^i f^r_{\mathbf{h},*}\mathcal{D}^r_{\widetilde{\mathfrak{Y}}^\wedge} &= Rg_* R\varprojlim_n R^i f^r_{\mathbf{h},*}\mathcal{D}^r_{\widetilde{\mathfrak{Y}}_n} = R\varprojlim_n Rg_* R^i f^r_{\mathbf{h},*}\mathcal{D}^r_{\widetilde{\mathfrak{Y}}_n} \\ &= R\varprojlim_n g_* R^i f^r_{\mathbf{h},*}\mathcal{D}^r_{\widetilde{\mathfrak{Y}}_n} = \varprojlim_n g_* R^i f^r_{\mathbf{h},*}\mathcal{D}^r_{\widetilde{\mathfrak{Y}}_n}. \end{aligned}$$

The last equality follows from that the exact functor g_* sends a Mittag-Leffler system to a Mittag-Leffler system.

(3) Suppose that $R^i f^r_{\mathbf{h},*}\mathcal{D}^r_{\widetilde{\mathfrak{Y}}^\wedge}$ is a vector bundle. The isomorphism $\varphi : \varphi^*\mathcal{D}^r_{\widetilde{\mathfrak{Y}}^\wedge} \simeq \mathcal{D}^r_{\widetilde{\mathfrak{Y}}^\wedge}|_{\mathbb{U}^{r/p}_{\widetilde{\mathfrak{Y}}^\wedge}}$ induces via $Rf^{r/p}_{\mathbf{h},*}$ isomorphisms $\varphi^* Rf^r_{\mathbf{h},*}\mathcal{D}^r_{\widetilde{\mathfrak{Y}}^\wedge} \simeq Rf^{r/p}_{\mathbf{h},*}(\mathcal{D}^r_{\widetilde{\mathfrak{Y}}^\wedge}|_{\mathbb{U}^{r/p}_{\widetilde{\mathfrak{Y}}^\wedge}}) = (Rf^r_{\mathbf{h},*}\mathcal{D}^r_{\widetilde{\mathfrak{Y}}^\wedge})|_{\mathbb{U}^{r/p}_{\mathfrak{Y}^\wedge}}$ (by flat base changes). By (3) of Lemma 5.2 and (2), $R^i f_{\mathbf{h},*} D^r_{\widetilde{\mathfrak{Y}}^\wedge}$ is a finite projective $D^r_{\mathfrak{Y}^\wedge}$-module which also ensures that $\mathcal{R}^{r'}_{\mathfrak{Y}^\wedge} \otimes_{\mathcal{R}^r_{\mathfrak{Y}^\wedge}} R^i f_{\mathbf{h},*} D^r_{\widetilde{\mathfrak{Y}}^\wedge}$ is finite projective corresponding to the restriction of $R^i f^r_{\mathbf{h},*}\mathcal{D}^r_{\widetilde{\mathfrak{Y}}^\wedge}$ to $\mathbb{U}^{r'}_{\mathfrak{Y}^\wedge}$. The isomorphism follows from that $\mathcal{R}^{r'}_{\widetilde{\mathfrak{Y}}^\wedge} \otimes_{\mathcal{R}^r_{\widetilde{\mathfrak{Y}}^\wedge}} D^r_{\widetilde{\mathfrak{Y}}^\wedge}$ corresponds to the restriction of $\mathcal{D}^r_{\widetilde{\mathfrak{Y}}^\wedge}$ to $\mathbb{U}^{r'}_{\widetilde{\mathfrak{Y}}^\wedge}$ via $(\widetilde{g}|_{\mathbb{U}^{r'}_{\widetilde{\mathfrak{Y}}^\wedge}})_*$ and the φ-structure on $R^i f_{\mathbf{h},*} D^r_{\widetilde{\mathfrak{Y}}^\wedge}$ comes from the φ-structure on $R^i f^r_{\mathbf{h},*}\mathcal{D}^r_{\widetilde{\mathfrak{Y}}^\wedge}$ via $(g|_{\mathbb{U}^{r/p}_{\mathfrak{Y}^\wedge}})_*$.

(4) The map $\widetilde{\mathfrak{Y}}_1 \to \mathfrak{Y}_1$ is a proper map between quasi-compact spaces, the direct image of $\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge}$-modules can be computed as $R^i f_{\mathbf{h},*} D_{\widetilde{\mathfrak{Y}}^\wedge} = R^i f_{\mathbf{h},*} \varinjlim_r D^r_{\widetilde{\mathfrak{Y}}^\wedge} = \varinjlim_r R^i f_{\mathbf{h},*} D^r_{\widetilde{\mathfrak{Y}}^\wedge}$ by [Sta24, Tag 0739, Tag 07TA] and [GV06, Exposé V, Prop. 5.1] (and [Bos14, Prop. 6.3/4]). The last equality follows from that colimits commute with tensor product. □

5.2. **Direct image and translation.** In this subsection we prove our main results on geometric translations of (φ,Γ)-modules in the Construction 5.4. Discussions in §3.4 are served for the following hypothesis.

Hypothesis 5.9. We assume that the map $\mathfrak{Y}^{\wedge,\square} := \varinjlim_n \mathfrak{Y}_n^{\square} \to \mathfrak{g}$ in Construction 5.4 is flat, in the sense that it satisfies the conclusion of Corollary 3.28 (and hence Remark 3.29).

Proposition 5.10. *Assume Hypothesis 5.9 and let* $\mathbf{h} = (h_1, h_2)$ *such that* $k = h_2 - h_1 \in \mathbb{Z}_{\geq 1}$. *Then* $Rf^r_{\mathbf{h},*}\mathcal{D}^r_{\widetilde{\mathfrak{Y}}^\wedge}$ *concentrates in degree zero and is a locally finite projective module of rank* 4 *over* $\mathcal{O}_{\mathbb{U}^r_{\widetilde{\mathfrak{Y}}^\wedge}}$.

Proof. By a twist, we assume that $h_1 = 0$. Then by construction, there are inclusions of vector bundles $t^k\Delta^r_{\widetilde{\mathfrak{Y}}_n} \hookrightarrow \mathcal{D}^r_{\widetilde{\mathfrak{Y}}_n} \hookrightarrow \Delta^r_{\widetilde{\mathfrak{Y}}_n}$.

Note that $\Delta^r_{\widetilde{\mathfrak{Y}}_n}/t^k\Delta^r_{\widetilde{\mathfrak{Y}}_n} = \prod_{m\geq m(r)} D^{m,+}_{\mathrm{dif}}(\Delta_{\widetilde{\mathfrak{Y}}_n})/t^k$ (cf. [Liu15, Prop. 2.15]) and taking inverse limit $\Delta^r_{\widetilde{\mathfrak{Y}}^\wedge}/t^k\Delta^r_{\widetilde{\mathfrak{Y}}^\wedge} = \prod_{m\geq m(r)} D^{m,+}_{\mathrm{dif}}(\Delta_{\widetilde{\mathfrak{Y}}^\wedge})/t^k$. The latter can be viewed as a coherent $\mathcal{O}_{\mathbb{U}^r_{\widetilde{\mathfrak{Y}}^\wedge}}$-module supported on disjoint divisors cut out by $Q_m(X)^k$ for $m \geq m(r)$, where $t = X\prod_{m\geq 1} Q_m(X)$, see Appendix A.1. Write $\mathbb{U}^{\wedge,m}$ for the completion of $\mathbb{U}^r$ along locus $\mathrm{Sp}(L\otimes_{\mathbb{Q}_p} K_m)$ cut out by $Q_m(X)$. The sheaf $D^{m,+}_{\mathrm{dif}}(\Delta_{\widetilde{\mathfrak{Y}}^\wedge})$ is exactly the $Q_m(X)$-adic completion of $\Delta^r_{\widetilde{\mathfrak{Y}}^\wedge}$ and we can identify $D^{m,+}_{\mathrm{dif}}(\Delta_{\widetilde{\mathfrak{Y}}^\wedge})/t$ as the pullback of $\Delta_{\widetilde{\mathfrak{Y}}^\wedge}$ to $\widetilde{\mathfrak{Y}}^\wedge \times_L \mathrm{Sp}(L\otimes_{\mathbb{Q}_p} K_m) \subset \widetilde{\mathfrak{Y}}^\wedge \times_L \mathbb{U}^r$.

On $\widetilde{\mathfrak{Y}}^\wedge$ the rank two projective $\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge}$-module $D_{\mathrm{pdR}} := D_{\mathrm{pdR}}(\Delta_{\widetilde{\mathfrak{Y}}^\wedge})$ is equipped with a universal submodule Fil^0 stabilized by $\nu_{\widetilde{\mathfrak{Y}}^\wedge}$ (now $\nu_{\widetilde{\mathfrak{Y}}^\wedge}$ is only topologically nilpotent). Set the decreasing filtration $\mathrm{Fil}^\bullet$ on $D_{\mathrm{pdR}}(\Delta_{\widetilde{\mathfrak{Y}}^\wedge})$ by $\mathrm{Fil}^{-k} = D_{\mathrm{pdR}} \supsetneq \mathrm{Fil}^{-k+1} = \cdots = \mathrm{Fil}^0 \supsetneq \mathrm{Fil}^1 = \{0\}$. Under the equivalence in Proposition A.6, for each m, the filtration $\mathrm{Fil}^\bullet$ gives the projective sub-$\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge\times_L\mathbb{U}^{\wedge,m}}$-module $D^{m,+}_{\mathrm{dif}}(D_{\widetilde{\mathfrak{Y}}^\wedge})$, a modification of $D^{m,+}_{\mathrm{dif}}(\Delta_{\widetilde{\mathfrak{Y}}^\wedge})$. By the identification in Step 1 of the proof of Proposition A.6 and the assumption that $m(r)$ is large enough (which holds modulo I^n and after taking inverse limit as well), as $\mathfrak{S}_{\widetilde{\mathfrak{Y}}^\wedge,m} := \mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge\times_L\mathbb{U}^{\wedge,m}} = (\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge}\times_{\mathbb{Q}_p} K_m)[[t]]$-modules

$$D^{m,+}_{\mathrm{dif}}(D_{\widetilde{\mathfrak{Y}}^\wedge}) = D_{\mathrm{pdR}} \otimes_{\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge}} t^k\mathfrak{S}_{\widetilde{\mathfrak{Y}}^\wedge,m} + \mathrm{Fil}^0 \otimes_{\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge}} \mathfrak{S}_{\widetilde{\mathfrak{Y}}^\wedge,m}$$

and

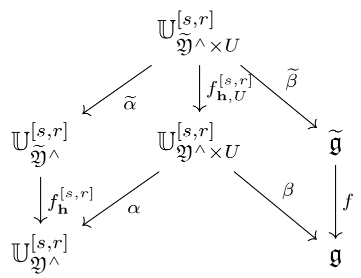
$$D^{m,+}_{\mathrm{dif}}(\Delta_{\widetilde{\mathfrak{Y}}^\wedge}) = D_{\mathrm{pdR}} \otimes_{\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge}} \mathfrak{S}_{\widetilde{\mathfrak{Y}}^\wedge,m}, D^{m,+}_{\mathrm{dif}}(t^k\Delta_{\widetilde{\mathfrak{Y}}^\wedge}) = D_{\mathrm{pdR}} \otimes_{\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge}} t^k\mathfrak{S}_{\widetilde{\mathfrak{Y}}^\wedge,m}. \tag{5.1}$$

Since the sequence $0 \to \mathrm{Fil}^0 \to D_{\mathrm{pdR}} \to D_{\mathrm{pdR}}/\mathrm{Fil}^0 \to 0$ splits locally as $\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge}$-modules, the injection

$$\mathrm{Fil}^0 \otimes_{\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge}} (\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge} \otimes_{\mathbb{Q}_p} K_m)[[t]]/t^k \to D^{m,+}_{\mathrm{dif}}(D_{\widetilde{\mathfrak{Y}}^\wedge})/t^k D^{m,+}_{\mathrm{dif}}(\Delta_{\widetilde{\mathfrak{Y}}^\wedge}) \tag{5.2}$$

is an isomorphism.

To show that $Rf^r_{\mathbf{h},*}\mathcal{D}^r_{\widetilde{\mathfrak{Y}}^\wedge}$ is locally free of rank 4, we can work locally on $\mathbb{U}^r_{\widetilde{\mathfrak{Y}}^\wedge}$. Take $s < r$. Write $\mathcal{D}^{[s,r]}_{\widetilde{\mathfrak{Y}}^\wedge}, \Delta^{[s,r]}_{\widetilde{\mathfrak{Y}}^\wedge}$ for their restriction to $\mathbb{U}^{[s,r]}_{\widetilde{\mathfrak{Y}}^\wedge}$ and $f^{[s,r]}_{\mathbf{h}} : \mathbb{U}^{[s,r]}_{\widetilde{\mathfrak{Y}}^\wedge} \to \mathbb{U}^{[s,r]}_{\mathfrak{Y}^\wedge}$. It's enough to show $Rf^{[s,r]}_{\mathbf{h},*}\mathcal{D}^{[s,r]}_{\widetilde{\mathfrak{Y}}^\wedge}$ is projective of rank 4 over $\mathcal{O}_{\mathbb{U}^{[s,r]}_{\mathfrak{Y}^\wedge}}$ for all r, s such that $m(s) = m(r)+1$. We may assume that the torsor $\mathfrak{Y}^{\wedge,\square} \to \mathfrak{Y}^\wedge$ is trivial as in Remark 3.29. Choose an affinoid open $U = \mathrm{Sp}(B) \subset \mathrm{GL}_2$ and consider the restriction $f^{[s,r]}_{\mathbf{h},U} : \mathbb{U}^{[s,r]}_{\widetilde{\mathfrak{Y}}^\wedge\times U} \to \mathbb{U}^{[s,r]}_{\mathfrak{Y}^\wedge\times U}$. We get the diagram



where the parallelograms are Cartesian (modulo I^n).

The formal rigid space $\mathbb{U}^{[s,r]}_{\mathfrak{Y}^\wedge}$ (resp. $\mathbb{U}^{[s,r]}_{\mathfrak{Y}^\wedge\times U}$) is the completion of an affinoid space and $\mathbb{U}^{[s,r]}_{\widetilde{\mathfrak{Y}}^\wedge}$ (resp. $\mathbb{U}^{[s,r]}_{\widetilde{\mathfrak{Y}}^\wedge\times U}$) comes from a projective scheme over $\mathrm{Spec}(\mathcal{O}(\mathbb{U}^{[s,r]}_{\mathfrak{Y}^\wedge}))$ (resp. $\mathrm{Spec}(\mathcal{O}(\mathbb{U}^{[s,r]}_{\mathfrak{Y}^\wedge\times U}))$) in the sense before Corollary B.6. Hence $R^i f^{[s,r]}_{\mathbf{h},*}\mathcal{D}^{[s,r]}_{\widetilde{\mathfrak{Y}}^\wedge}$ is a coherent $\mathcal{O}_{\mathbb{U}^{[s,r]}_{\mathfrak{Y}^\wedge}}$-module for all $i\geq 0$ and can be computed on the scheme level. The same statement holds similarly for $R^i f^{[s,r]}_{\mathbf{h},U,*}\widetilde{\alpha}^*\mathcal{D}^{[s,r]}_{\widetilde{\mathfrak{Y}}^\wedge}$. Also the maps $\mathbb{U}^{[s,r]}_{\mathfrak{Y}_n\times U}\to\mathfrak{g}\to\mathfrak{g}^{\mathrm{alg}}$ in Remark 3.29 factor through $\mathrm{Spec}(\mathcal{O}(\mathbb{U}^{[s,r]}_{\mathfrak{Y}_n\times U}))\to\mathfrak{g}^{\mathrm{alg}}$. Then $\alpha^*R^i f^{[s,r]}_{\mathbf{h},*}\mathcal{D}^{[s,r]}_{\widetilde{\mathfrak{Y}}^\wedge}=R^i f^{[s,r]}_{\mathbf{h},U,*}\widetilde{\alpha}^*\mathcal{D}_{\widetilde{\mathfrak{Y}}^\wedge}$ by the flat base change [Sta24, Tag 02KH] and $R^i f^{[s,r]}_{\mathbf{h},*}\mathcal{D}^{[s,r]}_{\widetilde{\mathfrak{Y}}^\wedge}$ is projective of rank 4 if and only if so is $R^i f^{[s,r]}_{\mathbf{h},U,*}\widetilde{\alpha}^*\mathcal{D}^{[s,r]}_{\widetilde{\mathfrak{Y}}^\wedge}=R^i f^{[s,r]}_{\mathbf{h},U,*}\mathcal{D}^{[s,r]}_{\widetilde{\mathfrak{Y}}^\wedge\times U}$ by the faithfully flat descent [Sta24, Tag 058S] and Lemma 5.11 below.

Consider the short exact sequences of coherent $\mathcal{O}_{\mathbb{U}^{[s,r]}_{\mathfrak{Y}^\wedge\times U}}$-modules from the first part of the proof (e.g., (5.2) and $m=m(r)$)

$$0\to t^k\Delta^{[s,r]}_{\widetilde{\mathfrak{Y}}^\wedge\times U}\to\mathcal{D}^{[s,r]}_{\widetilde{\mathfrak{Y}}^\wedge\times U}\to\mathrm{Fil}^0\otimes_{\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge}}(\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge\times U}\otimes_{\mathbb{Q}_p}K_m)[[t]]/t^k\to 0,$$
$$0\to\mathcal{D}^{[s,r]}_{\widetilde{\mathfrak{Y}}^\wedge\times U}\to\Delta^{[s,r]}_{\widetilde{\mathfrak{Y}}^\wedge\times U}\to D_{\mathrm{pdR}}/\mathrm{Fil}^0\otimes_{\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge}}(\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge\times U}\otimes_{\mathbb{Q}_p}K_m)[[t]]/t^k\to 0.$$

By our construction of $\widetilde{\mathfrak{Y}}^{\wedge,\square}$, the pullback $D_{\mathrm{pdR}}\otimes_{\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge}}\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge\times U}$ admits a tautological trivialization $D_{\mathrm{pdR}}\otimes_{\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge}}\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge\times U}\simeq\mathcal{O}^{\oplus 2}_{\widetilde{\mathfrak{Y}}^\wedge\times U}$ and the subsheaf $\mathrm{Fil}^0\otimes_{\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge}}\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge\times U}=\widetilde{D}^*_{\mathrm{pdR}}\mathcal{O}_{\widetilde{\mathfrak{g}}}(-1)$ is the tautological subbundle pulled back from $\widetilde{\mathfrak{g}}\to G/B=\mathbf{P}^1$ where we write D_{pdR} for the flat map $\mathrm{Spec}(\mathcal{O}(\mathfrak{Y}^\wedge\times U))\to\mathfrak{g}$ (induced by the nilpotent operator ν) and $\widetilde{D}_{\mathrm{pdR}}$ for the base change of D_{pdR}.

Let $f_{\mathbf{h},U}:\widetilde{\mathfrak{Y}}^\wedge\times U\to\mathfrak{Y}^\wedge\times U$ be the base change of $f_{\mathbf{h}}$. The sheaf $\mathrm{Fil}^0\otimes_{\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge}}(\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge\times U}\otimes_{\mathbb{Q}_p}K_m)$ is supported on the closed subscheme of $\mathbb{U}^{[s,r]}_{\widetilde{\mathfrak{Y}}^\wedge\times U}$ cut out by $Q_m(X)$ and

$$Rf^{[s,r]}_{\mathbf{h},U,*}\mathrm{Fil}^0\otimes_{\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge}}(\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge\times U}\otimes_{\mathbb{Q}_p}K_m)[[t]]/t^k=(Rf_{\mathbf{h},U,*}\widetilde{D}^*_{\mathrm{pdR}}\mathcal{O}_{\widetilde{\mathfrak{g}}}(-1))\otimes_{\mathbb{Q}_p}K_m[[t]]/t^k$$

as a coherent sheaf supported on the closed subscheme of $\mathbb{U}^{[s,r]}_{\mathfrak{Y}^\wedge\times U}$ cut out by $Q_m(X)^k$. By the flat base change and (1) of Proposition 2.4, $Rf_{\mathbf{h},U,*}\widetilde{D}^*_{\mathrm{pdR}}\mathcal{O}_{\widetilde{\mathfrak{g}}}(\pm 1)=D^*_{\mathrm{pdR}}Rf_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(\pm 1)$ concentrate in degree zero and are locally free of rank two over $\mathfrak{Y}^\wedge\times U$.

By the projection formula, $Rf^{[s,r]}_{\mathbf{h},U,*}\Delta^{[s,r]}_{\widetilde{\mathfrak{Y}}^\wedge\times U}=\Delta^{[s,r]}_{\mathfrak{Y}^\wedge\times U}\otimes_{\mathcal{O}_{\mathbb{U}^{[s,r]}_{\mathfrak{Y}^\wedge\times U}}}Rf^{[s,r]}_{\mathbf{h},U,*}Rf^{[s,r],*}_{\mathbf{h},U}\mathcal{O}_{\mathbb{U}^{[s,r]}_{\mathfrak{Y}^\wedge\times U}}$. Since β is flat by the assumption that the map $\mathfrak{Y}^{\wedge,\square}\to\mathfrak{g}$ is flat, $Rf^{[s,r]}_{\mathbf{h},U,*}Rf^{[s,r],*}_{\mathbf{h},U}\mathcal{O}_{\mathbb{U}^{[s,r]}_{\mathfrak{Y}^\wedge\times U}}=\beta^*Rf_*Rf^*\mathcal{O}_{\mathfrak{g}}=\beta^*f_*\mathcal{O}_{\widetilde{\mathfrak{g}}}$ is free of rank two over $\mathcal{O}_{\mathbb{U}^{[s,r]}_{\mathfrak{Y}^\wedge\times U}}$. Then we get short exact sequences

$$0\to f^{[s,r]}_{\mathbf{h},U,*}t^k\Delta^{[s,r]}_{\widetilde{\mathfrak{Y}}^\wedge\times U}\to f^{[s,r]}_{\mathbf{h},U,*}\mathcal{D}^{[s,r]}_{\widetilde{\mathfrak{Y}}^\wedge\times U}\to D^*_{\mathrm{pdR}}f_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(-1)\otimes_{\mathbb{Q}_p}K_m[[t]]/t^k\to 0,\tag{5.3}$$
$$0\to f^{[s,r]}_{\mathbf{h},U,*}\mathcal{D}^{[s,r]}_{\widetilde{\mathfrak{Y}}^\wedge\times U}\to f^{[s,r]}_{\mathbf{h},U,*}\Delta^{[s,r]}_{\widetilde{\mathfrak{Y}}^\wedge\times U}\to D^*_{\mathrm{pdR}}f_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(1)\otimes_{\mathbb{Q}_p}K_m[[t]]/t^k\to 0.$$

Finally to see that $f^{[s,r]}_{\mathbf{h},U,*}\mathcal{D}^{[s,r]}_{\widetilde{\mathfrak{Y}}^\wedge\times U}$ is locally free of rank 4 over $\mathcal{O}_{\mathbb{U}^{[s,r]}_{\mathfrak{Y}^\wedge\times U}}$, by [BL95], it is enough to consider its completion along the divisor $(\widetilde{\mathfrak{Y}}^\wedge\times_L U)\times_L\mathrm{Sp}(L\otimes_{\mathbb{Q}_p}K_m)$ cut out by $Q_m(X)$, which is a $(\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge\times U}\times_{\mathbb{Q}_p}K_m)[[t]]$-module. Then the result follows from Lemma 3.17, that $f^{[s,r]}_{\mathbf{h},U,*}\Delta^{[s,r]}_{\widetilde{\mathfrak{Y}}^\wedge\times U}$ is free of rank 4 over $\mathcal{O}_{\mathbb{U}^{[s,r]}_{\mathfrak{Y}^\wedge\times U}}$ and that $D^*_{\mathrm{pdR}}f_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(1)\otimes_{\mathbb{Q}_p}K_m[[t]]/t^k$ is finite flat over $\mathcal{O}_{\mathfrak{Y}^\wedge\times U}\times_{\mathbb{Q}_p}K_m[[t]]/t^k$. □

Lemma 5.11. *Let $g:X=\mathrm{Sp}(B)\to Y=\mathrm{Sp}(A)$ be a flat (resp. faithfully flat) morphism of affinoid spaces and let $I\subset A$ be an ideal. Let $A^\wedge,B^\wedge$ be the I-adic completions. Then the ring maps $A\to B$ and $A^\wedge\to B^\wedge$ are flat (resp. faithfully flat).*

Proof. By definition, for any $x\in X$ and $f(x)\in Y$ corresponding to maximal ideals $\mathfrak{m}\subset B,\mathfrak{n}\subset A$, the local ring map $\mathcal{O}_{Y,f(x)}\to\mathcal{O}_{X,y}$ is flat. Since both rings are Noetherian, flatness can be checked after completion and thus also on the Zariski local rings $A_{\mathfrak{n}}\to B_{\mathfrak{m}}$. By [Sta24, Tag 00HT], B is flat over A. The assertion about flatness after completion is by Lemma B.7. For faithfully flatness, if $\mathrm{Spec}(B^\wedge)\to\mathrm{Spec}(A^\wedge)$ is surjective on closed points and flat, it is surjective by [Sta24, Tag 00HS].

As I is topologically nilpotent in $A^\wedge$ and $A^\wedge$ is I-adically complete, I is in the Jacobson radical of $A^\wedge$. The surjectivity on closed points follows from the surjectivity of $\mathrm{Spec}(B/I) \to \mathrm{Spec}(A/I)$. $\square$

Over $\widetilde{\mathfrak{Y}}^\wedge$ the topologically nilpotent operator $\nu_{\mathfrak{Y}^\wedge}$ acts on the universal sub-line bundle Fil^0 of $D_{\mathrm{pdR}}(\Delta_{\widetilde{\mathfrak{Y}}^\wedge})$ and the quotient $D_{\mathrm{pdR}}(\Delta_{\widetilde{\mathfrak{Y}}^\wedge})/\mathrm{Fil}^0$. This gives elements $z_{\widetilde{\mathfrak{Y}}^\wedge} + h_{\widetilde{\mathfrak{Y}}^\wedge} \in \mathcal{O}(\widetilde{\mathfrak{Y}}^\wedge) \simeq \mathrm{End}_{\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge}}(\mathrm{Fil}^0)$ and $z_{\widetilde{\mathfrak{Y}}^\wedge} - h_{\widetilde{\mathfrak{Y}}^\wedge} \in \mathcal{O}(\widetilde{\mathfrak{Y}}^\wedge) \simeq \mathrm{End}_{\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge}}(D_{\mathrm{pdR}}(\Delta_{\widetilde{\mathfrak{Y}}^\wedge})/\mathrm{Fil}^0)$. We view $z_{\widetilde{\mathfrak{Y}}^\wedge}$ and $h_{\widetilde{\mathfrak{Y}}^\wedge}$ as global sections of $f_{\mathbf{h},*}\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge}$ (cf. Corollary B.6). As a corollary of the proof of Proposition 5.10 and (3) of Proposition 2.4, we get the following explicit description of $Rf_{\mathbf{h},*}D^r_{\widetilde{\mathfrak{Y}}^\wedge}$ when $\mathbf{h} = (0,k)$.

Corollary 5.12. *Suppose* $\mathbf{h} = (0,k), k \geq 1$ *and assume Hypothesis 5.9. Then* $Rf_{\mathbf{h},*}\Delta^r_{\widetilde{\mathfrak{Y}}^\wedge} = \Delta^r_{\mathfrak{Y}^\wedge} \otimes_{\mathcal{O}_{\mathfrak{Y}^\wedge}} f_{\mathbf{h},*}\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge}$ *is a* (φ,Γ)*-module of rank* 4 *over* $\mathcal{R}^r_{\mathfrak{Y}^\wedge}$. *And* $Rf_{\mathbf{h},*}D^r_{\widetilde{\mathfrak{Y}}^\wedge}$ *is the rank 4 sub-*(φ,Γ)*-module of* $f_{\mathbf{h},*}\Delta^r_{\widetilde{\mathfrak{Y}}^\wedge}$ *containing* $t^k f_{\mathbf{h},*}\Delta^r_{\widetilde{\mathfrak{Y}}^\wedge}$ *determined by*

$$f_{\mathbf{h},*}D^r_{\widetilde{\mathfrak{Y}}^\wedge}/t^k f_{\mathbf{h},*}\Delta^r_{\widetilde{\mathfrak{Y}}^\wedge} = \oplus_{i=0}^{k-1}(f_{\mathbf{h},*}\Delta^r_{\widetilde{\mathfrak{Y}}^\wedge}/t^k f_{\mathbf{h},*}\Delta^r_{\widetilde{\mathfrak{Y}}^\wedge})[\nabla_{\mathrm{Sen}} = (z_{\widetilde{\mathfrak{Y}}^\wedge} + h_{\widetilde{\mathfrak{Y}}^\wedge}) + i]. \tag{5.4}$$

Proof. We write $\widetilde{\Delta}^r_{\mathfrak{Y}^\wedge}$ for $\Delta^r_{\mathfrak{Y}^\wedge} \otimes_{\mathcal{O}_{\mathfrak{Y}^\wedge}} f_{\mathbf{h},*}\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge}$. By Proposition 5.10, the map $f_{\mathbf{h},*}D^r_{\widetilde{\mathfrak{Y}}^\wedge} \to \widetilde{\Delta}^r_{\mathfrak{Y}^\wedge}$ induced by $D^r_{\widetilde{\mathfrak{Y}}^\wedge} \subset \Delta^r_{\widetilde{\mathfrak{Y}}^\wedge}$ is injective and the image contains $t^k\widetilde{\Delta}^r_{\mathfrak{Y}^\wedge}$. Hence it suffices to determine the image of

$$f_{\mathbf{h},*}D^r_{\widetilde{\mathfrak{Y}}^\wedge}/t^k\widetilde{\Delta}^r_{\mathfrak{Y}^\wedge} \hookrightarrow \widetilde{\Delta}^r_{\mathfrak{Y}^\wedge}/t^k\widetilde{\Delta}^r_{\mathfrak{Y}^\wedge} = \prod_{m\geq m(r)} D^{m,+}_{\mathrm{dif}}(\widetilde{\Delta}_{\mathfrak{Y}^\wedge})/t^k. \tag{5.5}$$

The operator ∇_{Sen} acts on each $D^{m,+}_{\mathrm{dif}}(\widetilde{\Delta}_{\mathfrak{Y}^\wedge})/t^k$. Under the identification (5.1), $D_{\mathrm{pdR}}(\widetilde{\Delta}_{\mathfrak{Y}^\wedge}) \otimes_{\mathbb{Q}_p} K_m[[t]]/t^k = (D_{\mathrm{pdR}}(\Delta_{\mathfrak{Y}^\wedge}) \otimes_{\mathbb{Q}_p} K_m[[t]]/t^k) \otimes_{\mathcal{O}_{\mathfrak{Y}^\wedge}} f_{\mathbf{h},*}\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge}$, and the Sen operator ∇_{Sen} corresponds to the topologically nilpotent operator $\nu_{\mathfrak{Y}^\wedge}$ on $D_{\mathrm{pdR}}(\Delta_{\mathfrak{Y}^\wedge})$ where we extend the action of $\nu_{\mathfrak{Y}^\wedge}$ on $v \otimes g \in D_{\mathrm{pdR}}(\Delta_{\mathfrak{Y}^\wedge}) \otimes_{A^\wedge} (A^\wedge \otimes_{\mathbb{Q}_p} K_m)[[t]]$ by $\nu_{\mathfrak{Y}^\wedge}(v \otimes g) = \nu_{\mathfrak{Y}^\wedge}(v) \otimes g + v \otimes \nabla(g)$ (see Step 1 of the proof of Proposition A.6, and note $\nabla_{\mathrm{Sen}}(tx) = t(\nabla_{\mathrm{Sen}}+1)(x)$). The identity (5.4) will follow from that the image of (5.5) is equal to the submodule

$$\prod_{m\geq m(r)} \oplus_{i=0}^{k-1}(D_{\mathrm{pdR}}(\widetilde{\Delta}_{\mathfrak{Y}^\wedge}) \otimes_{\mathbb{Q}_p} K_m[[t]]/t^k)[\nu_{\mathfrak{Y}^\wedge} = (z_{\widetilde{\mathfrak{Y}}^\wedge} + h_{\widetilde{\mathfrak{Y}}^\wedge} + i)].$$

We view objects appeared in (5.5) as coherent sheaves on $\mathbb{U}^r_{\mathfrak{Y}^\wedge}$ supported on divisors cut out by $Q_m(X)$ for $m \geq m(r)$. By faithfully flat descent, we only need to verify the equality after base change to $\mathfrak{Y}^\wedge \times U$ as in the proof of Proposition 5.10 and we adapt the notation there. By (5.3), we see

$$f_{\mathbf{h},U,*}D^r_{\widetilde{\mathfrak{Y}}^\wedge\times U}/t^k\widetilde{\Delta}^r_{\mathfrak{Y}^\wedge\times U} = \prod_{m\geq m(r)} D^*_{\mathrm{pdR}} f_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(-1) \otimes_{\mathbb{Q}_p} K_m[[t]]/t^k.$$

By (3) of Proposition 2.4, the righthand side is the sheaf

$$\prod_{m\geq m(r)} D^*_{\mathrm{pdR}}((f_*f^*(\mathcal{O}_{\mathfrak{g}}^{\oplus 2}))[\nu = (h+z)]) \otimes_{\mathbb{Q}_p} K_m[[t]]/t^k.$$

Since D_{pdR} is flat, $D^*_{\mathrm{pdR}} f_*f^*(\mathcal{O}_{\mathfrak{g}}^{\oplus 2}) = f_{\mathbf{h},U,*}f^*_{\mathbf{h},U}D_{\mathrm{pdR}}(\Delta_{\mathfrak{Y}^\wedge\times U}) = D_{\mathrm{pdR}}(\widetilde{\Delta}_{\mathfrak{Y}^\wedge\times U})$ (under the canonical trivialization $D_{\mathrm{pdR}}(\Delta_{\mathfrak{Y}^\wedge\times U}) \simeq D^*_{\mathrm{pdR}}\mathcal{O}_{\mathfrak{g}}^{\oplus 2}$) and that ν, z, and h are pulled back to $\nu_{\mathfrak{Y}^\wedge\times U}$, $z_{\widetilde{\mathfrak{Y}}^\wedge}$, and $h_{\widetilde{\mathfrak{Y}}^\wedge}$ respectively, we get

$$\begin{aligned} f_{\mathbf{h},U,*}D^r_{\widetilde{\mathfrak{Y}}^\wedge\times U}/t^k\widetilde{\Delta}^r_{\mathfrak{Y}^\wedge\times U} &= \prod_{m\geq m(r)} D_{\mathrm{pdR}}(\widetilde{\Delta}_{\mathfrak{Y}^\wedge\times U})[\nu_{\mathfrak{Y}^\wedge\times U} = (h_{\widetilde{\mathfrak{Y}}^\wedge} + z_{\widetilde{\mathfrak{Y}}^\wedge})] \otimes_{\mathbb{Q}_p} K_m[[t]]/t^k \\ &= \prod_{m\geq m(r)} \oplus_{i=0}^{k-1}(t^i D_{\mathrm{pdR}}(\widetilde{\Delta}_{\mathfrak{Y}^\wedge\times U}) \otimes_{\mathbb{Q}_p} K_m)[\nu_{\mathfrak{Y}^\wedge\times U} = (h_{\widetilde{\mathfrak{Y}}^\wedge} + z_{\widetilde{\mathfrak{Y}}^\wedge}) + i] \end{aligned}$$

Hence the description (5.4) holds. $\square$

We define translations of formal completions of (φ,Γ)-modules.

Definition 5.13. Let $\mathfrak{X}^\wedge = \varinjlim_n \mathfrak{X}_n$ be the formal completion of a quasi-compact rigid space $\mathfrak{X}$ with respect to a coherent ideal sheaf $\mathcal{I}$ (Definition B.1). Let $D^r_{\mathfrak{X}^\wedge}$ be a $(\varphi,\Gamma,\mathfrak{g})$-module over $\mathcal{R}^r_{\mathfrak{X}^\wedge}$ (Definition 5.1). Assume that $D^r_{\mathfrak{X}_1}$ is locally $Z(\mathfrak{g})$-finite (in the sense that there is a finite admissible covering of $\mathfrak{X}_1$ by open affinoids such that $D^r_{\mathfrak{X}_1}$ is locally $Z(\mathfrak{g})$-finite when restricted to these affinoids). Let $\lambda,\mu\in X^*(\mathfrak{t})$ be integral weights. We define the translation of $D^r_{\mathfrak{X}^\wedge}$ from the infinitesimal character associated to λ to that of μ by

$$T^\mu_\lambda D^r_{\mathfrak{X}^\wedge} := \varprojlim_n T^\mu_\lambda D^r_{\mathfrak{X}_n}.$$

And if $D_{\mathfrak{X}^\wedge}=\mathcal{R}_{\mathfrak{X}^\wedge}\otimes_{\mathcal{R}^r_{\mathfrak{X}^\wedge}} D^r_{\mathfrak{X}^\wedge}$,

$$T^\mu_\lambda D_{\mathfrak{X}^\wedge} := \varinjlim_{r'\le r} T^\mu_\lambda D^{r'}_{\mathfrak{X}^\wedge}.$$

The translation is always a (φ,Γ)-module with a $\mathfrak{g}$-action (cf. Proposition 4.12).

Recall that if $\lambda-\mu$ is $(0,k)$ or $(k,0)$ for some $k\ge 0$, then $T^\mu_\lambda D^r_{\mathfrak{X}_n}=\mathrm{pr}_{|\mu|}(\mathrm{pr}_{|\lambda|}D^r_{\mathfrak{X}_n}\otimes_L V_k)$.

Lemma 5.14. *Let $f_{\mathbf{h}}:\widetilde{\mathfrak{Y}}^\wedge\to\mathfrak{Y}^\wedge$ be the map in Construction 5.4. Suppose that $D^{',r}_{\widetilde{\mathfrak{Y}}^\wedge}$ is a $(\varphi,\Gamma,\mathfrak{g})$-module over $\mathcal{R}^r_{\widetilde{\mathfrak{Y}}^\wedge}$ such that $Rf_{\mathbf{h},*}D^{',r}_{\widetilde{\mathfrak{Y}}^\wedge}$ is a (φ,Γ)-module over $\mathfrak{Y}^\wedge$ (cf. Lemma 5.8).*

(1) The natural isomorphism $Rf_{\mathbf{h},*}(D^{',r}_{\widetilde{\mathfrak{Y}}^\wedge}\otimes_L V_k)\simeq(Rf_{\mathbf{h},*}D^{',r}_{\widetilde{\mathfrak{Y}}^\wedge})\otimes_L V_k$ of $\mathcal{O}_{\mathfrak{Y}^\wedge}$-modules (provided by $V_k\simeq L^{\oplus(k+1)}$) is an isomorphism of $(\varphi,\Gamma,\mathfrak{g})$-modules.

(2) Furthermore, if $D^{',r}_{\widetilde{\mathfrak{Y}}_1}$ is locally $Z(\mathfrak{g})$-finite with a generalized infinitesimal character given by λ, then the isomorphism in (1) induces an isomorphism $Rf_{\mathbf{h},*}T^\mu_\lambda D^{',r}_{\widetilde{\mathfrak{Y}}^\wedge}\simeq T^\mu_\lambda Rf_{\mathbf{h},*}D^{',r}_{\widetilde{\mathfrak{Y}}^\wedge}$.

Proof. (1) We verify the isomorphism $f_{\mathbf{h},*}(D^{',r}_{\widetilde{\mathfrak{Y}}^\wedge}\otimes_L V_k)\simeq(f_{\mathbf{h},*}D^{',r}_{\widetilde{\mathfrak{Y}}^\wedge})\otimes_L V_k$ is an isomorphism of $(\varphi,\Gamma,\mathfrak{g})$-module. Recall $V^k=\mathrm{Sym}^kL^2=\mathcal{R}^+_L/X^{k+1}$. Then $D^{',r}_{\widetilde{\mathfrak{Y}}^\wedge}\otimes_L V_k=\oplus_{i=0}^k D^{',r}_{\widetilde{\mathfrak{Y}}^\wedge}\otimes_L X^iL$ and $f_{\mathbf{h},*}(D^{',r}_{\widetilde{\mathfrak{Y}}^\wedge}\otimes_L V_k)=\oplus_{i=0}^k f_{\mathbf{h},*}D^{',r}_{\widetilde{\mathfrak{Y}}^\wedge}\otimes_L X^iL\simeq f_{\mathbf{h},*}D^{',r}_{\widetilde{\mathfrak{Y}}^\wedge}\otimes_L V_k$ with obvious maps. And the sheaf $f_{\mathbf{h},*}(D^{',r}_{\widetilde{\mathfrak{Y}}^\wedge}\otimes_L V_k)$ is determined by its section over $\mathfrak{Y}^\wedge$ as for $f_{\mathbf{h},*}D^{',r}_{\widetilde{\mathfrak{Y}}^\wedge}$ (Lemma 5.2). For $g\in\mathfrak{g}$ and $\sum_{i=0}^k a_i\otimes X^i\in f_{\mathbf{h},*}D^{',r}_{\widetilde{\mathfrak{Y}}^\wedge}\otimes_L V_k$, we have $g.(\sum_{i=0}^k a_i\otimes X^i)=\sum_{i=0}^k g.a_i\otimes X^i+\sum_{i=0}^k a_i\otimes g.X^i=\sum_{i=0}^k(g.a_i+\sum_j c_{ji}a_j)\otimes X^i$ where $g.X^i=\sum_{i,j}c_{ij}X^j$. The $\mathfrak{g}$ action on $\sum_{i=0}^k a_i\otimes X^i\in f_{\mathbf{h},*}(D^{',r}_{\widetilde{\mathfrak{Y}}^\wedge}\otimes_L V_k)$ (resp. on $a_i\in f_{\mathbf{h},*}D^{',r}_{\widetilde{\mathfrak{Y}}^\wedge}$) is given by the same formula viewing $\sum_{i=0}^k a_i\otimes X^i$ (resp. a_i) as global sections on $\widetilde{\mathfrak{Y}}^\wedge$. Hence the $\mathfrak{g}$-actions coincide. Similar statements hold for the actions of φ and Γ. It remains to show that the map is $\mathcal{R}^r_{\mathfrak{Y}^\wedge}$-linear. By Lemma 5.8, we have $f_{\mathbf{h},*}(D^{',r}_{\widetilde{\mathfrak{Y}}^\wedge}\otimes_L V_k)=\varprojlim_n f_{\mathbf{h},*}(D^{',r}_{\widetilde{\mathfrak{Y}}_n}\otimes_L V_k)$ and $f_{\mathbf{h},*}D^{',r}_{\widetilde{\mathfrak{Y}}^\wedge}=\varprojlim_n f_{\mathbf{h},*}D^{',r}_{\widetilde{\mathfrak{Y}}_n}$ as sheaves of $\mathcal{R}^r_{\mathfrak{Y}^\wedge}$-module. Let $f=(f_n)_n\in\mathcal{R}^r_{\mathfrak{Y}^\wedge}=\varprojlim_n\mathcal{R}^r_{\mathfrak{Y}_n}$ act on $\varprojlim_n f_{\mathbf{h},*}D^{',r}_{\widetilde{\mathfrak{Y}}_n}$. For $\sum_{i=0}^k a_i\otimes X^i\in f_{\mathbf{h},*}D^{',r}_{\widetilde{\mathfrak{Y}}^\wedge}\otimes_L V_k$, write similarly $a_i=(a_{i,n})_n\in\varprojlim_n f_{\mathbf{h},*}D^{',r}_{\widetilde{\mathfrak{Y}}_n}$. By the proof of Proposition 4.6, the action of f_n on $a_{i,n}\otimes X^i$ is given by $f_n.(a_{i,n}\otimes X^i)=\sum_{j=0}^k\frac{1}{j!}(X+1)^jf_n^{(j)}(X)a_{i,n}\otimes X^{j+i}$. Same equation holds if we view $a_{i,n}\otimes X^i$ as a global section of $D^{',r}_{\widetilde{\mathfrak{Y}}_n}\otimes_L V_k$ for the action of f_n via $f_{\mathbf{h}}^{-1}\mathcal{R}^r_{\mathfrak{Y}_n}\to\mathcal{R}^r_{\widetilde{\mathfrak{Y}}_n}$. Taking inverse limit we get the compatibility of $\mathcal{R}^r_{\mathfrak{Y}^\wedge}$-actions.

(2) If $D^{',r}_{\widetilde{\mathfrak{Y}}_n}$ is locally $Z(\mathfrak{g})$-finite, since $\widetilde{\mathfrak{Y}}_n$ is quasi-compact, $R^if_{\mathbf{h},*}D^{',r}_{\widetilde{\mathfrak{Y}}_n}$ is locally $Z(\mathfrak{g})$-finite (one can choose a finite affinoid covering of $\widetilde{\mathfrak{Y}}_n$ to calculate the cohomology). By taking inverse limit, we have a direct sum decomposition $D^{',r}_{\widetilde{\mathfrak{Y}}^\wedge}\otimes_L V_k=\oplus_{\mu'}T^{\mu'}_\lambda D^{',r}_{\widetilde{\mathfrak{Y}}^\wedge}$ for finitely many μ'. We conclude by noticing that $Z(\mathfrak{g})$ acts on $f_{\mathbf{h},*}T^\mu_\lambda D^{',r}_{\widetilde{\mathfrak{Y}}^\wedge}=\varprojlim_n f_{\mathbf{h},*}T^\mu_\lambda D^{',r}_{\widetilde{\mathfrak{Y}}_n}$ locally profinitely with generalized infinitesimal character μ. □

Theorem 5.15. *Let $\lambda=\lambda_{\mathbf{h}}=(h_2-1,h_1)$ for $\mathbf{h}=(h_1,h_2)\in\mathbb{Z}^2,h_1<h_2$ and $\mu=\lambda_{(0,0)}=(-1,0)$ be weights in $X^*(\mathfrak{t})$. Let $f_{\mathbf{h}}:\widetilde{\mathfrak{Y}}^\wedge\to\mathfrak{Y}^\wedge$ and $D^r_{\widetilde{\mathfrak{Y}}^\wedge},\Delta^r_{\mathfrak{Y}^\wedge}$ be the map and (φ,Γ)-modules in Construction 5.4. We equip $D^r_{\widetilde{\mathfrak{Y}}^\wedge}$ and $\Delta^r_{\mathfrak{Y}^\wedge}$ with the standard $\mathfrak{g}$-module structures in Definition 4.4.*

(1) There is an isomorphism of $(\varphi,\Gamma,\mathfrak{g})$-modules

$$T^\mu_\lambda D^r_{\widetilde{\mathfrak{Y}}^\wedge}\simeq f^*_{\mathbf{h}}\Delta_{\mathfrak{Y}^\wedge}:=\Delta^r_{\widetilde{\mathfrak{Y}}^\wedge}=\varprojlim_n\Delta^r_{\widetilde{\mathfrak{Y}}_n}\tag{5.6}$$

over $\mathcal{R}^r_{\widetilde{\mathfrak{Y}}^\wedge}$.

(2) *Under Hypothesis 5.9, the adjunction of* $f_{\mathbf{h}}^* T_\mu^\lambda \Delta^r_{\mathfrak{Y}^\wedge} = T_\mu^\lambda \Delta^r_{\widetilde{\mathfrak{Y}}^\wedge} \to D^r_{\widetilde{\mathfrak{Y}}^\wedge}$ *(as* (φ,Γ)*-bundles over* $\mathbb{U}^r_{\widetilde{\mathfrak{Y}}^\wedge}$*) induces an isomorphism*

$$T_\mu^\lambda \Delta^r_{\mathfrak{Y}^\wedge} \simeq Rf_{\mathbf{h},*} D^r_{\widetilde{\mathfrak{Y}}^\wedge} \tag{5.7}$$

of $(\varphi,\Gamma,\mathfrak{g})$*-modules of rank* 4 *over* $\mathcal{R}^r_{\mathfrak{Y}^\wedge}$.

Proof. (1) The statement follows from the construction of $D^r_{\widetilde{\mathfrak{Y}}^\wedge} = \varprojlim_n D^r_{\widetilde{\mathfrak{Y}}_n}, T_\lambda^\mu D^r_{\widetilde{\mathfrak{Y}}^\wedge} = \varprojlim_n T_\lambda^\mu D^r_{\widetilde{\mathfrak{Y}}_n}$ and Proposition 4.17.

(2) We first assume that $\mathbf{h} = (0,1)$. By Proposition 5.10 and Hypothesis 5.9, $Rf_{\mathbf{h},*} D^r_{\widetilde{\mathfrak{Y}}^\wedge} = f_{\mathbf{h},*} D^r_{\widetilde{\mathfrak{Y}}^\wedge}$. Write $\widetilde{\Delta}^r_{\mathfrak{Y}^\wedge}$ for $\Delta^r_{\mathfrak{Y}^\wedge} \otimes_{\mathcal{O}_{\mathfrak{Y}^\wedge}} f_{\mathbf{h},*}\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge}$. The unit map for (φ,Γ)-bundles and the projection formula induce a map $T_\mu^\lambda \Delta^r_{\mathfrak{Y}^\wedge} \to T_\mu^\lambda \Delta^r_{\mathfrak{Y}^\wedge} \otimes_{\mathcal{O}_{\mathfrak{Y}^\wedge}} f_{\mathbf{h},*} f_{\mathbf{h}}^* \mathcal{O}_{\mathfrak{Y}^\wedge} = T_\mu^\lambda \widetilde{\Delta}^r_{\mathfrak{Y}^\wedge} = f_{\mathbf{h},*} T_\mu^\lambda \Delta^r_{\widetilde{\mathfrak{Y}}^\wedge}$ (the last equation is by Lemma 5.14). The isomorphism $\Delta^r_{\widetilde{\mathfrak{Y}}^\wedge} \simeq T_\lambda^\mu D^r_{\widetilde{\mathfrak{Y}}^\wedge}$ induces $T_\mu^\lambda \Delta^r_{\widetilde{\mathfrak{Y}}^\wedge} \to D^r_{\widetilde{\mathfrak{Y}}^\wedge}$ and hence $T_\mu^\lambda f_{\mathbf{h},*} \Delta^r_{\widetilde{\mathfrak{Y}}^\wedge} \to f_{\mathbf{h},*} D^r_{\widetilde{\mathfrak{Y}}^\wedge}$. The composite of the two maps gives the desired $\mathfrak{g}$-map

$$T_\mu^\lambda \Delta^r_{\mathfrak{Y}^\wedge} \to f_{\mathbf{h},*} T_\mu^\lambda \Delta^r_{\widetilde{\mathfrak{Y}}^\wedge} \to f_{\mathbf{h},*} D^r_{\widetilde{\mathfrak{Y}}^\wedge}.$$

We show that this map is an isomorphism of (φ,Γ)-modules over $\mathcal{R}^r_{\mathfrak{Y}^\wedge}$.

By Hypothesis 5.9, flat base change and faithfully flat descent as in the proof of Proposition 5.10, statements of Proposition 2.4 hold replacing $f : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ by $f_{\mathbf{h}} : \widetilde{\mathfrak{Y}} \to \mathfrak{Y}$ after suitable modifications. In particular, $f_{\mathbf{h},*}\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge}$ is locally free of rank two over $\mathcal{O}_{\mathfrak{Y}^\wedge}$ generated by the element $h_{\widetilde{\mathfrak{Y}}^\wedge}$ defined before Corollary 5.12. Write for short $z = z_{\widetilde{\mathfrak{Y}}^\wedge}$ and $h = h_{\widetilde{\mathfrak{Y}}^\wedge}$. The composite $T_\mu^\lambda \Delta^r_{\mathfrak{Y}^\wedge} \to T_\mu^\lambda \widetilde{\Delta}^r_{\mathfrak{Y}^\wedge} \overset{\frac14 \mathfrak{c} - h^2 - h}{\to} T_\mu^\lambda \widetilde{\Delta}^r_{\mathfrak{Y}^\wedge} \twoheadrightarrow \widetilde{\Delta}^r_{\mathfrak{Y}^\wedge}$ sends $v_0 \otimes 1 + v_1 \otimes t \in T_\mu^\lambda \Delta^r_{\mathfrak{Y}^\wedge}$ to $(-\nabla + z - h)v_0 + tv_1$ by the proof of Proposition 4.20 (all these maps are inverse limits of maps modulo I^n). This map is an injection ($\widetilde{\Delta}_{\mathfrak{Y}^\wedge} = \Delta_{\mathfrak{Y}^\wedge} \oplus h\Delta_{\mathfrak{Y}^\wedge}$) and the image contains $t\widetilde{\Delta}_{\mathfrak{Y}^\wedge}$ (for $thv_0 \in th\Delta_{\mathfrak{Y}^\wedge}$, let $v_1 = \frac{(\nabla - z)tv_0}{t}$). Modulo $t\widetilde{\Delta}^r_{\mathfrak{Y}^\wedge}$, the image of $T_\mu^\lambda \Delta^r_{\mathfrak{Y}^\wedge}$ as a sub-$\mathcal{O}_{\mathfrak{Y}^\wedge}$-module of $\widetilde{\Delta}^r_{\mathfrak{Y}^\wedge}/t = \prod_{m \geq m(r)} D^m_{\mathrm{Sen}}(\widetilde{\Delta}_{\mathfrak{Y}^\wedge}) \otimes_{\mathbb{Q}_p} K_m$ is $\prod_{m \geq m(r)} (\nabla - (z-h)) D^m_{\mathrm{Sen}}(\Delta_{\mathfrak{Y}^\wedge}) \otimes_{\mathbb{Q}_p} K_m$. By (2) of Proposition 2.4 and the flatness Hypothesis 5.9, this image is identified with $\prod_{m \geq m(r)} D^m_{\mathrm{Sen}}(\widetilde{\Delta}_{\mathfrak{Y}^\wedge})[\nabla = (z+h)] \otimes_{\mathbb{Q}_p} K_m$, i.e., is equal to $f_{\mathbf{h},*} D^r_{\widetilde{\mathfrak{Y}}^\wedge}/t\widetilde{\Delta}^r_{\mathfrak{Y}^\wedge}$ by Corollary 5.12. By Proposition 4.20 and taking direct image, the composite $f_{\mathbf{h},*} T_\mu^\lambda \Delta^r_{\widetilde{\mathfrak{Y}}^\wedge} \to f_{\mathbf{h},*} D^r_{\widetilde{\mathfrak{Y}}^\wedge} \hookrightarrow f_{\mathbf{h},*} \Delta^r_{\widetilde{\mathfrak{Y}}^\wedge}$ is identified with $T_\mu^\lambda \widetilde{\Delta}^r_{\mathfrak{Y}^\wedge} \overset{\frac14 \mathfrak{c} - h^2 - h}{\to} T_\mu^\lambda \widetilde{\Delta}^r_{\mathfrak{Y}^\wedge} \twoheadrightarrow \widetilde{\Delta}^r_{\mathfrak{Y}^\wedge}$. Hence the composite $T_\mu^\lambda \Delta^r_{\mathfrak{Y}^\wedge} \to f_{\mathbf{h},*} T_\mu^\lambda \Delta^r_{\widetilde{\mathfrak{Y}}^\wedge} \to f_{\mathbf{h},*} D^r_{\widetilde{\mathfrak{Y}}^\wedge}$ is an isomorphism, with the same image in $\widetilde{\Delta}^r_{\mathfrak{Y}^\wedge}$ given by Corollary 5.12. We have finished the case $\mathbf{h} = (0,1)$.

Finally, we treat general $\mathbf{h}' = (0,k), k \geq 1$. Write $\lambda' = \lambda_{\mathbf{h}'}$. We have simply $f_{\mathbf{h}} = f_{\mathbf{h}'} : \widetilde{\mathfrak{Y}}^\wedge \to \mathfrak{Y}^\wedge$. Write $D'^{,r}_{\widetilde{\mathfrak{Y}}^\wedge}$ for the universal (φ,Γ)-module over $\mathcal{R}^r_{\widetilde{\mathfrak{Y}}^\wedge}$ of weights $\mathbf{h}'$ and write $D^r_{\widetilde{\mathfrak{Y}}^\wedge}$ for the one with weights $\mathbf{h} = (0,1)$. By Proposition 4.17, $D'^{,r}_{\widetilde{\mathfrak{Y}}^\wedge} \simeq T_\lambda^{\lambda'} D^r_{\widetilde{\mathfrak{Y}}^\wedge}$. Then $f_{\mathbf{h}',*} D'^{,r}_{\widetilde{\mathfrak{Y}}^\wedge} \simeq f_{\mathbf{h}',*} T_\lambda^{\lambda'} D^r_{\widetilde{\mathfrak{Y}}^\wedge} \simeq T_\lambda^{\lambda'} f_{\mathbf{h}',*} D^r_{\widetilde{\mathfrak{Y}}^\wedge}$ by Lemma 5.14. Since $T_\mu^\lambda \Delta^r_{\mathfrak{Y}^\wedge} \simeq f_{\mathbf{h},*} D^r_{\widetilde{\mathfrak{Y}}^\wedge}$, we see $f_{\mathbf{h}',*} D'^{,r}_{\widetilde{\mathfrak{Y}}^\wedge} \simeq T_\lambda^{\lambda'} T_\mu^\lambda \Delta^r_{\mathfrak{Y}^\wedge} \simeq T_\mu^{\lambda'} \Delta^r_{\mathfrak{Y}^\wedge}$ (Lemma 4.10). Note that this isomorphism is induced by $T_\mu^\lambda \Delta^r_{\mathfrak{Y}^\wedge} \otimes_L V_{k-1} \simeq f_{\mathbf{h},*} D^r_{\widetilde{\mathfrak{Y}}^\wedge} \otimes_L V_{k-1}$ which, by the case for $\mathbf{h}$, is induced by the adjunction of $T_\mu^\lambda \Delta^r_{\widetilde{\mathfrak{Y}}^\wedge} \otimes_L V_{k-1} \to D^r_{\widetilde{\mathfrak{Y}}^\wedge} \otimes_L V_{k-1}$. Taking $\mathrm{pr}_{|\lambda'|}$ everywhere we see the isomorphism $f_{\mathbf{h}',*} D'^{,r}_{\widetilde{\mathfrak{Y}}^\wedge} \simeq T_\mu^{\lambda'} \Delta^r_{\mathfrak{Y}^\wedge}$ is induced by $T_\mu^{\lambda'} \Delta^r_{\widetilde{\mathfrak{Y}}^\wedge} \to D'^{,r}_{\widetilde{\mathfrak{Y}}^\wedge}$. □

Taking colimit for $r > 0$ and considering Lemma 5.8, we get the following corollary.

Corollary 5.16. *Under Hypothesis 5.9, we have*

$$T_\lambda^\mu D_{\widetilde{\mathfrak{Y}}^\wedge} \simeq f_{\mathbf{h}}^* \Delta_{\mathfrak{Y}^\wedge}$$

and

$$T_\mu^\lambda \Delta_{\mathfrak{Y}^\wedge} \simeq Rf_{\mathbf{h},*} D_{\widetilde{\mathfrak{Y}}^\wedge}.$$

5.3. **Specialization to points.** We show how to use Theorem 5.15 to recover some results of Ding. Suppose we are in the situation of Construction 5.4. We fix an L-point y of $\mathfrak{Y}^\wedge$ and write $\Delta = \Delta_y$ for the specialization of $\Delta_{\mathfrak{Y}^\wedge}$ at the point y. Let $i_y : y \hookrightarrow \mathfrak{Y}_1 \hookrightarrow \mathfrak{Y}^\wedge$ be the closed embedding and consider the restriction $f_y : f_{\mathbf{h}}^{-1}(y) \to y$ of $f_{\mathbf{h}}$.

$$\begin{array}{ccc} f_{\mathbf{h}}^{-1}(y) & \xhookrightarrow{i'_y} & \widetilde{\mathfrak{Y}}^{\wedge} \\ \Big\downarrow{\scriptstyle f_y} & & \Big\downarrow{\scriptstyle f_{\mathbf{h}}} \\ y & \xhookrightarrow{i_y} & \mathfrak{Y}^{\wedge} \end{array}$$

By the projection formula [Sta24, Tag 08EU] (and the arguments in the proof of Proposition 5.10 of reducing to scheme-theoretic cohomologies), we have

$$i_{y,*}Li_y^*Rf_{\mathbf{h},*}D^r_{\widetilde{\mathfrak{Y}}^{\wedge}} = Rf_{\mathbf{h},*}D^r_{\widetilde{\mathfrak{Y}}^{\wedge}} \otimes^L_{\mathcal{O}_{\mathfrak{Y}}} i_{y,*}\mathcal{O}_y \simeq Rf_{\mathbf{h},*}(D^r_{\widetilde{\mathfrak{Y}}^{\wedge}} \otimes^L_{\mathcal{O}_{\widetilde{\mathfrak{Y}}^{\wedge}}} Lf_{\mathbf{h}}^* i_{y,*}\mathcal{O}_y). \tag{5.8}$$

By Theorem 5.15, the left-hand side equals $T_\mu^\lambda \Delta_y^r$ assuming Hypothesis 5.9.

Lemma 5.17. *Assume Hypothesis 5.9.*

1. *If Δ_y is de Rham, then $f_{\mathbf{h}}^{-1}(y) = G/B = \mathbf{P}^1$ is reduced and*
$$Lf_{\mathbf{h}}^* i_{y,*}\mathcal{O}_y = i'_{y,*}(\mathcal{O}_{\mathbf{P}^1}(-2)[1] \oplus \mathcal{O}_{\mathbf{P}^1}[0])$$
where $\mathcal{O}_{\mathbf{P}^1}(-2)[1]$ denotes the line bundle $\mathcal{O}_{\mathbf{P}^1}(-2)$ sitting in cohomological degree -1.
2. *If Δ_y is not de Rham, then $f_{\mathbf{h}}^{-1}(y)$ is a finite ramified cover of degree 2 over y and $Lf_{\mathbf{h}}^* i_{y,*}\mathcal{O}_y = f_{\mathbf{h}}^* i_{y,*}\mathcal{O}_y = i'_{y,*}\mathcal{O}_{f_{\mathbf{h}}^{-1}(y)}$.*

Proof. Consider the diagram (of schemes, to be lazy) below:

$$\begin{array}{ccccccc} f_{\mathbf{h}}^{-1}(y) & \xhookrightarrow{i'_y} & \widetilde{\mathfrak{Y}}^{\wedge} & \xleftarrow[\widetilde{\alpha}]{} & \widetilde{\mathfrak{Y}}^{\wedge,\square} & \xrightarrow{\widetilde{\beta}} & \widetilde{\mathfrak{g}} \\ \Big\downarrow{\scriptstyle f_y} & & \Big\downarrow{\scriptstyle f_{\mathbf{h}}} & & \Big\downarrow{\scriptstyle f_{\mathbf{h}}^{\square}} & & \Big\downarrow{\scriptstyle f} \\ y & \xhookrightarrow{i_y} & \mathfrak{Y}^{\wedge} & \xleftarrow[\alpha]{} & \mathfrak{Y}^{\wedge,\square} & \xrightarrow{\beta} & \mathfrak{g} \end{array}$$

where all squares are Cartesian. Choose a lift $y^\square$ of y in $\mathfrak{Y}^{\wedge,\square}$ and let ν_y be the image of y in $\mathfrak{g}$. If the nilpotent element $\nu_y \neq 0$, all vertical maps are finite flat of rank 2 near $y, y^\square$ or ν_y by the statement on $\mathfrak{g}$ (Lemma 2.3) and we get (2).

From now on we assume $\nu_y = 0$ and prove (1). Consider the embedding $\widetilde{\mathfrak{Y}}^{\wedge,\square} \xrightarrow{j} H := \mathfrak{Y}^{\wedge,\square} \times G/B$ and still write $\beta : \mathfrak{Y}^{\wedge,\square} \times G/B \to \mathfrak{g} \times G/B$. Let $\mathcal{I}_0$ be the ideal sheaf for the regular closed embedding $\widetilde{\mathfrak{g}} \hookrightarrow \mathfrak{g} \times G/B$, which is locally free of rank one. Since β is flat, $\mathcal{I} := \beta^*\mathcal{I}_0$ is the ideal sheaf cutting out $\widetilde{\mathfrak{Y}}^{\wedge,\square}$ from $\mathfrak{Y}^{\wedge,\square} \times G/B$. Let $\mathcal{J}$ be the ideal sheaf for the closed embedding $\alpha^{-1}(y) \times G/B \hookrightarrow \mathfrak{Y}^{\wedge,\square} \times G/B$. Then $\mathcal{I} \subset \mathcal{J}$ as points on $\alpha^{-1}(y)$ are all de Rham ($\alpha^{-1}(y) \times G/B \hookrightarrow \widetilde{\mathfrak{Y}}^{\wedge,\square}$).

$$\begin{array}{ccccc} f_{\mathbf{h}}^{\square,-1}(\alpha^{-1}(y)) = \alpha^{-1}(y) \times G/B & \longrightarrow & \widetilde{\mathfrak{Y}}^{\wedge,\square} & \xhookrightarrow{j} & H := \mathfrak{Y}^{\wedge,\square} \times G/B \\ \Big\downarrow{\scriptstyle f_{\mathbf{h}}^{\square}} & & \Big\downarrow{\scriptstyle f_{\mathbf{h}}^{\square}} & \swarrow{\scriptstyle k} & \\ \alpha^{-1}(y) & \xrightarrow{i_{\alpha^{-1}(y)}} & \mathfrak{Y}^{\wedge,\square} & & \end{array}$$

In the notation of the above diagram, using that k is smooth, we get

$$Lf_{\mathbf{h}}^{\square,*} i_{\alpha^{-1}(y),*}\mathcal{O}_{\alpha^{-1}(y)} = Lj^*k^*i_{\alpha^{-1}(y),*}\mathcal{O}_{\alpha^{-1}(y)} = Lj^*\mathcal{O}_H/\mathcal{J}.$$

While

$$Lj^*\mathcal{O}_H/\mathcal{J} = \mathcal{O}_H/\mathcal{I} \otimes^L_{\mathcal{O}_H} \mathcal{O}_H/\mathcal{J} = [\mathcal{I} \to \mathcal{O}_H] \otimes_{\mathcal{O}_H} \mathcal{O}_H/\mathcal{J} = [\mathcal{I} \otimes_{\mathcal{O}_H} \mathcal{O}_H/\mathcal{J} \to \mathcal{O}_H/\mathcal{J}].$$

Since $\mathcal{I} \subset \mathcal{J}$, the map $\mathcal{I} \otimes_{\mathcal{O}_H} \mathcal{O}_H/\mathcal{J} \to \mathcal{O}_H/\mathcal{J}$ is zero. We calculate the restriction of the line bundle $\mathcal{I}$ to $\alpha^{-1}(y) \times G/B$ whose structure sheaf is $\mathcal{O}_H/\mathcal{J}$. The ideal $\mathcal{I}_0$ on $\mathfrak{g} \times G/B$ corresponds to the condition that $(\nu, gB) \in \mathfrak{g} \times G/B$ such that $\mathrm{Ad}(g^{-1})(\nu) \in \mathfrak{b}$. Write $\mathrm{Ad}(g^{-1})(\nu) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then change g to gh for $h = \begin{pmatrix} x & y \\ & x^{-1} \end{pmatrix} \in B$ sends a local generator $c \in \mathcal{I}_0$ to x^2c. This means that the restriction of $\mathcal{I}_0$ to G/B is $\mathcal{O}_{\mathbf{P}^1}(-2)$ ($G \times^B \lambda$ for $\lambda = (1, -1)$). Thus $Lf_{\mathbf{h}}^{\square,*} i_{\alpha^{-1}(y),*}\mathcal{O}_{\alpha^{-1}(y)} = \mathcal{O}_{f_{\mathbf{h}}^{\square,-1}(\alpha^{-1}(y))} \oplus \mathcal{O}_{f_{\mathbf{h}}^{\square,-1}(\alpha^{-1}(y))}(-2)[1]$. The statement for $Lf_{\mathbf{h}}^* i_{y,*}\mathcal{O}_y$ follows from descent. $\square$

Remark 5.18. Write $i_0 : \{0\} \hookrightarrow \mathfrak{g}$ and $\mathbf{P}^1 = f^{-1}(0)$. With the proof of the lemma above we see $Li_0^* Rf_*\mathcal{O}_{\widetilde{\mathfrak{g}}} = R(f|_{\mathbf{P}^1})_*(\mathcal{O}_{\mathbf{P}^1}(-2)[1] \oplus \mathcal{O}_{\mathbf{P}^1}[0]) = \mathcal{O}^2_{\{0\}}$. The calculation matches the fact that $Rf_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(s)$ is locally free of rank two for $s = 0, \pm 1$ in §2.

Write $D_{\mathbf{P}^1}$ for the universal (φ,Γ)-module on $f_{\mathbf{h}}^{-1}(y)$ (the restriction of $D_{\mathfrak{Y}_1}$ to $f_{\mathbf{h}}^{-1}(y)$).

Lemma 5.19. *If* $\mathbf{h} = (0,1)$, *then* $Rf_{y,*}(D^r_{\mathbf{P}^1}) = t\Delta^r$ *and* $Rf_{y,*}(D^r_{\mathbf{P}^1}(-2)[1]) = \Delta^r$.

Proof. The proof goes as for Proposition 5.10 and Corollary 5.12. We only do formal calculations here. Let $\Delta_{\mathbf{P}^1}$ be the pullback of Δ. The inclusions $t\Delta^r_{\mathbf{P}^1} \subset D^r_{\mathbf{P}^1} \subset \Delta^r_{\mathbf{P}^1}$ gives short exact sequences

$$0 \to t\Delta^r_{\mathbf{P}^1} \to D^r_{\mathbf{P}^1} \to D^r_{\mathbf{P}^1}/t\Delta^r_{\mathbf{P}^1} \to 0$$

and

$$0 \to D^r_{\mathbf{P}^1} \to \Delta^r_{\mathbf{P}^1} \to D^r_{\mathbf{P}^1}/\Delta^r_{\mathbf{P}^1} \to 0.$$

The inclusion $D^r_{\mathbf{P}^1}/t\Delta^r_{\mathbf{P}^1} \hookrightarrow \Delta^r_{\mathbf{P}^1}/t\Delta^r_{\mathbf{P}^1} = \mathcal{R}^r_L/t\widehat{\otimes}_L\mathcal{O}^2_{\mathbf{P}^1}$ has image $\mathcal{R}^r_L/t\widehat{\otimes}_L\mathcal{O}_{\mathbf{P}^1}(-1)$ and the quotient $\Delta^r_{\mathbf{P}^1}/t\Delta^r_{\mathbf{P}^1} \twoheadrightarrow \Delta^r_{\mathbf{P}^1}/D^r_{\mathbf{P}^1}$ corresponds to the quotient $\mathcal{R}^r_L/t\widehat{\otimes}_L\mathcal{O}^2_{\mathbf{P}^1} \twoheadrightarrow \mathcal{R}^r_L/t\widehat{\otimes}_L\mathcal{O}_{\mathbf{P}^1}(1)$. Hence we have

$$Rf_{y,*}(D^r_{\mathbf{P}^1}/t\Delta^r_{\mathbf{P}^1}) = Rf_{y,*}(\mathcal{R}^r_L/t\widehat{\otimes}_L\mathcal{O}_{\mathbf{P}^1}(-1)) = \mathcal{R}^r_L/t\widehat{\otimes}_L Rf_{y,*}\mathcal{O}_{\mathbf{P}^1}(-1) = 0,$$
$$Rf_{y,*}(\Delta^r_{\mathbf{P}^1}/D^r_{\mathbf{P}^1}(-2)) = Rf_{y,*}(\mathcal{R}^r_L/t\widehat{\otimes}_L\mathcal{O}_{\mathbf{P}^1}(-1)) = \mathcal{R}^r_L/t\widehat{\otimes}_L Rf_{y,*}\mathcal{O}_{\mathbf{P}^1}(-1) = 0.$$

Moreover,

$$Rf_{y,*}t\Delta^r_{\mathbf{P}^1} = t\Delta^r\widehat{\otimes}_L Rf_{y,*}\mathcal{O}_{\mathbf{P}^1} = t\Delta^r;$$
$$Rf_{y,*}(\Delta^r_{\mathbf{P}^1}(-2)[1]) = \Delta^r\widehat{\otimes}_L Rf_{y,*}\mathcal{O}_{\mathbf{P}^1}(-2)[1] = \Delta^r.$$

The result follows. □

We recover [Din23, Lem. 2.17] below.

Proposition 5.20. *Suppose that* $\mathbf{h} = (0,1)$ *and assume Hypothesis 5.9.*

(1) If $\Delta = \Delta_y$ *is de Rham, then* $T^\lambda_\mu\Delta = i_y^* f_{\mathbf{h},*}D_{\widetilde{\mathfrak{Y}}^\wedge} = \Delta \oplus t\Delta$.

(2) If $\Delta = \Delta_y$ *is not de Rham, then* $T^\lambda_\mu\Delta = i_y^* f_{\mathbf{h},*}D_{\widetilde{\mathfrak{Y}}^\wedge}$ *is a self extension of* D_y, *where* D_y *is the unique* (φ,Γ)*-module of rank two over* $\mathcal{R}_L$ *of weight* $\mathbf{h}$ *such that* $T^\mu_\lambda D_y = \Delta$.

Proof. Since $D^r_{\widetilde{\mathfrak{Y}}^\wedge}$ is flat over $\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge}$, we get $D^r_{\widetilde{\mathfrak{Y}}^\wedge} \otimes^L_{\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge}} Lf^*_{\mathbf{h}} i_{y,*}\mathcal{O}_y = i'_{y,*}(D^r_{\mathbf{P}^1}(-2)[1] \oplus D^r_{\mathbf{P}^1})$ if Δ_y is de Rham and is equal to $D_{f_{\mathbf{h}}^{-1}(y)}$ otherwise by Lemma 5.17. By Lemma 5.19, we have $Rf_{\mathbf{h},*}(D^r_{\widetilde{\mathfrak{Y}}^\wedge} \otimes^L_{\mathcal{O}_{\widetilde{\mathfrak{Y}}^\wedge}} Lf^*_{\mathbf{h}} i_{y,*}\mathcal{O}_y) = i_{y,*}Rf_{y,*}(D^r_{\mathbf{P}^1}(-2)[1] \oplus D^r_{\mathbf{P}^1}) = \Delta^r \oplus t\Delta^r$ in the de Rham case. Use (5.8) and take the direct limit over r, we get (1). (2) follows similarly using that $\mathcal{O}_{f_{\mathbf{h}}^{-1}(y)} \simeq L[h]/h^2$. Note that by Proposition 4.20 and the proof of Theorem 5.15, the Casimir $\mathfrak{c}$ acts by $4h^2 - 4h$ on $T^\lambda_\mu\Delta$ for some choice of h. □

Remark 5.21. In general for $\mathbf{h} = (0,k)$ and in the de Rham case, there is a filtration $t^k\Delta_{\mathbf{P}^1} \subset D_{(k-1,k)} \subset \cdots \subset D_{(1,k)} \subset D_{(0,k)} = D_{\mathbf{P}^1}$ of (φ,Γ)-modules of rank two where $D_{(i,k)}$ has Hodge-Tate weight (i,k) by the proof of Proposition 4.17. The graded pieces $D_{(i,k)}/D_{(i+1,k)}$ are isomorphic to $\mathcal{O}_{\mathbf{P}^1}(-1)\widehat{\otimes}_L\mathcal{R}_L/t$ as in Lemma 5.19 (see (5.2)) and we can get similarly $T^\lambda_\mu\Delta = \Delta \oplus t^k\Delta$.

5.4. **Translation of** $D \boxtimes \mathbf{P}^1(\mathbb{Q}_p)$**.** Let A be an affinoid algebra over L and D_A be a (φ,Γ)-module over $\mathcal{R}_A$ of rank 2. Write $\omega = \omega_A$ be the character such that $\mathcal{R}_A(\omega_A\epsilon) = \det(D_A)$. Pointwisely for $x \in \mathrm{Sp}(A)$, Colmez constructed a $\mathrm{GL}_2(\mathbb{Q}_p)$-representation (or a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant sheaf on $\mathbf{P}^1(\mathbb{Q}_p)$) $D_x \boxtimes_\omega \mathbf{P}^1(\mathbb{Q}_p)$ ([Col16, Col18, Col10]). It is expected that the construction can vary in family and obtain a $\mathrm{GL}_2(\mathbb{Q}_p)$-module $D_A \boxtimes_\omega \mathbf{P}^1(\mathbb{Q}_p)$. We will not discuss the $\mathrm{GL}_2(\mathbb{Q}_p)$-module, but only construct a $U(\mathfrak{g})$-module $D_A \boxtimes \mathbf{P}^1(\mathbb{Q}_p)$. Recall $\mathbf{P}^1(\mathbb{Q}_p) \setminus \mathbb{Z}_p = \Pi.\mathbb{Z}_p$ where $\Pi = \begin{pmatrix} & 1 \\ p & \end{pmatrix} = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} p & \\ & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_p)$.

Definition 5.22. Equip D_A with the standard $\mathfrak{g}$-module structure in Lemma 4.3. Define $D_A \boxtimes \mathbb{Z}_p = D_A$ to be the $U(\mathfrak{g})\otimes_L A$-module D_A. Let $D_A \boxtimes (\mathbf{P}^1(\mathbb{Q}_p)\setminus\mathbb{Z}_p)$ be the $U(\mathfrak{g})\otimes_L A$-module $\Pi.D_A$ which has the underlying A-module D_A and $\mathfrak{g}$ acts on $\Pi x, x \in D_A$ by $g.\Pi x = \Pi(\mathrm{Ad}(\Pi^{-1})(g).x)$. Define $D_A \boxtimes \mathbf{P}^1(\mathbb{Q}_p) = D_A \boxtimes \mathbb{Z}_p \oplus D_A \boxtimes (\mathbf{P}^1(\mathbb{Q}_p) \setminus \mathbb{Z}_p)$.

Remark 5.23. The character ω is not important for the $U(\mathfrak{g})$-modules due to our definition. But it matters for $\mathrm{GL}_2(\mathbb{Q}_p)$-representations, namely for how $\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ acts on $D \boxtimes \mathbb{Z}_p^\times$.

Hence $D_A \boxtimes \mathbf{P}^1(\mathbb{Q}_p)$ is just two copies of D_A with certain $\mathfrak{g}$-action. We can similarly define $D_A^r \boxtimes \mathbf{P}^1(\mathbb{Q}_p)$ $(= D_A^r \oplus \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \varphi(D_A^r))$. Let us go back to the situation in Construction 5.4. Sheafify the above definition we obtain sheaves of $U(\mathfrak{g})$-modules $D_{\widetilde{\mathfrak{Y}}^\wedge} \boxtimes \mathbf{P}^1(\mathbb{Q}_p)$ and $\Delta_{\mathfrak{Y}^\wedge} \boxtimes \mathbf{P}^1(\mathbb{Q}_p)$.

Corollary 5.24. *Under Hypothesis 5.9, there are isomorphisms of sheaves of $U(\mathfrak{g})$-modules*

$$T_\lambda^\mu(D_{\widetilde{\mathfrak{Y}}^\wedge} \boxtimes \mathbf{P}^1(\mathbb{Q}_p)) \simeq Rf_{\mathbf{h}}^*(\Delta_{\mathfrak{Y}^\wedge} \boxtimes \mathbf{P}^1(\mathbb{Q}_p))$$

and

$$T_\mu^\lambda(\Delta_{\mathfrak{Y}^\wedge} \boxtimes \mathbf{P}^1(\mathbb{Q}_p)) \simeq Rf_{\mathbf{h},*}(D_{\widetilde{\mathfrak{Y}}^\wedge} \boxtimes \mathbf{P}^1(\mathbb{Q}_p)),$$

where all notations appeared should be understood as for sheaves of (φ, Γ)-modules with certain $\mathfrak{g}$-actions.

Proof. Over $\mathbb{Z}_p$, this is just Corollary 5.16. For the copy on $\mathbf{P}^1(\mathbb{Q}_p) \setminus \mathbb{Z}_p$, just notice that for $k \geq 1$, $(\Pi.D_A) \otimes_L \mathrm{Sym}^k L^2 \simeq \Pi.(D_A \otimes_L \mathrm{Sym}^k L^2)$ as $U(\mathfrak{g})$-modules (since the $\mathfrak{g}$-action on $\mathrm{Sym}^k L^2$ integrates to a $\mathrm{GL}_2(\mathbb{Q}_p)$-action) and the adjoint action of Π acts trivially on the center of $U(\mathfrak{g})$. □

Appendix A. On families of (φ, Γ_K)-modules

A.1. **Beauville-Laszlo glueing.** We let K be a p-adic local field. Recall

$$t = \log([\epsilon]) = X_{\mathbb{Q}_p} \prod_{m \geq 1} Q_m(X_{\mathbb{Q}_p})/p.$$

Here Q_m is the minimal polynomial of $\zeta_{p^m} - 1$ over $\mathbb{Q}_p$, $X_{\mathbb{Q}_p} = [\epsilon] - 1$, $\epsilon = (1, \zeta_p, \cdots, \zeta_{p^m}, \cdots) \in \mathcal{O}^\flat_{\widehat{K}_\infty}$ where $\widehat{K}_\infty$ denotes the p-adic completion and ζ_{p^m} is a primitive p^m-th root of unity. For $m \geq m(r)$, the continuous Γ_K-equivariant injection $\iota_m : \mathcal{R}^r_{\mathbb{Q}_p,K} \hookrightarrow K_m[[t]]$ can be seen as the completion with respect to the kernel $(Q_m(X_{\mathbb{Q}_p}))$ of $\mathcal{R}^r_{\mathbb{Q}_p,K} \to K_m = \mathcal{R}^r_{\mathbb{Q}_p,K}/Q_m(X_{\mathbb{Q}_p})$ (see [Ber08a, §1.2] and [Ber02, Lem. 4.9]). For an affinoid algebra A over $\mathbb{Q}_p$, the ring $(A \otimes_{\mathbb{Q}_p} K_m)[[t]]$ is the completion of $\mathcal{R}^r_{A,K}$ with respect to the ideal $(Q_m(X_{\mathbb{Q}_p}))$ and we still write $\iota_m : \mathcal{R}^r_{A,K} \to (A \otimes_{\mathbb{Q}_p} K_m)[[t]]$.

Suppose that D_A is a (φ, Γ_K)-module over $\mathcal{R}_{A,K}$ of rank n, base changed from a (φ, Γ_K)-module D_A^r from $\mathcal{R}^r_{A,K}$ for some r. Define $D^{m,+}_{\mathrm{dif}}(D_A) := (A \widehat{\otimes} K_m)[[t]] \otimes_{\iota_m, \mathcal{R}^r_{A,K}} D_A^r$ for $m \geq m(r)$. Since by definition $\iota_{m+1} = \iota_m \circ \varphi^{-1}$, the map $K_m[[t]] \to K_{m+1}[[t]]$ induced by $\varphi : \mathcal{R}^r_K \to \mathcal{R}^{r/p}_K$ is $K_m[[t]]$-linear and $\varphi^* D_A^r = \mathcal{R}^{r/p}_{A,K} \otimes_{\varphi, \mathcal{R}^r_{A,K}} D_A^r \simeq D_A^{r/p}$ induces a Γ_K-linear isomorphism $(A \otimes_{\mathbb{Q}_p} K_{m+1})[[t]] \otimes_{(A \otimes_{\mathbb{Q}_p} K_m)[[t]]} D^{m,+}_{\mathrm{dif}}(D_A) \simeq D^{m+1,+}_{\mathrm{dif}}(D_A)$.

Definition A.1. A (φ, Γ_K)-module M_A (resp. M_A^r) over $\mathcal{R}_{A,K}[\frac{1}{t}]$ (resp. $\mathcal{R}^r_{A,K}[\frac{1}{t}]$) is a finite projective $\mathcal{R}_{A,K}[\frac{1}{t}]$-module (resp. finite projective $\mathcal{R}^r_{A,K}[\frac{1}{t}]$-module) equipped with commuting continuous semi-linear actions of φ, Γ_K (resp. Γ_K-isomorphism $\varphi^* M_A^r \simeq M_A^{r/p}$) such that there exists a (φ, Γ_K)-module D_A^r over $\mathcal{R}_A^r$ and an isomorphism $M_A \simeq \mathcal{R}_{A,K}[\frac{1}{t}] \otimes_{\mathcal{R}^r_{A,K}} D_A^r$ (resp. $M_A^r \simeq \mathcal{R}^r_{A,K}[\frac{1}{t}] \otimes_{\mathcal{R}^r_{A,K}} D_A^r$).

Lemma A.2. *The ring $\mathcal{R}^r_{A,K}$ is t torsion-free.*

Proof. The short exact sequence $0 \to \mathcal{R}_K^{[s,r]} \xrightarrow{\times t} \mathcal{R}_K^{[s,r]} \to \mathcal{R}_K^{[s,r]}/t \to 0$ splits as $\mathbb{Q}_p$-Banach spaces as in the proof of [Liu15, Prop. 2.15]. Taking the completed tensor with A we see $\mathcal{R}^{[s,r]}_{A,K}$ is t-torsion free for all r, s. The limit $\mathcal{R}^r_{A,K} = \varprojlim_s \mathcal{R}^{[s,r]}_{A,K}$ is still t-torsion free. □

The functor $D^m_{\mathrm{dif}}(-)$ extends naturally for (φ, Γ_K)-modules over $\mathcal{R}^r_{A,K}[\frac{1}{t}]$ and $m > m(r)$ by inverting t. And the φ-action induces $D^{m+1}_{\mathrm{dif}}(M_A) \simeq (A \otimes_{\mathbb{Q}_p} K_{m+1})[[t]][\frac{1}{t}] \otimes_{(A \otimes_{\mathbb{Q}_p} K_m)[[t]][\frac{1}{t}]} D^m_{\mathrm{dif}}(M_A)$.

Proposition A.3. *Suppose that r is chosen such that t is invertible in $\mathcal{R}^{[r,r]}_{A,K}$ and $m = m(r)$. The functor*

$$D_A^r \mapsto (M_A^r := D_A^r[\frac{1}{t}], D^{m,+}_{\mathrm{dif}}(D_A^r), D^m_{\mathrm{dif}}(M_A^r) = D^{m,+}_{\mathrm{dif}}(D_A^r)[\frac{1}{t}])$$

induces an equivalence of categories

$$\Phi\Gamma^{r,+}_{A,K} \simeq \Phi\Gamma^{r}_{A,K} \times_{\mathrm{Rep}^{m}_{\mathrm{dif},A}(\Gamma_K)} \mathrm{Rep}^{m,+}_{\mathrm{dif},A}(\Gamma_K)$$

between the category of (φ,Γ_K)-modules over $\mathcal{R}^r_{A,K}$ and the category of triples $(M^r_A, D^{m,+}_{\mathrm{dif},A}, \alpha_m)$ where M^r_A is a (φ,Γ_K)-module over $\mathcal{R}^r_{A,K}[\frac{1}{t}]$, $D^{m,+}_{\mathrm{dif},A}$ is a continuous semilinear Γ_K-representation over a projective $(A\otimes_{\mathbb{Q}_p} K_m)[[t]]$-module of rank n and $\alpha_m : D^{m,+}_{\mathrm{dif},A}[\frac{1}{t}] \simeq D^m_{\mathrm{dif}}(M^r_A)$. The inverse functor is given by (write $D^{m',+}_{\mathrm{dif},A} := (K_{m'}\otimes_{\mathbb{Q}_p} A)[[t]]\otimes_{(K_m\otimes_{\mathbb{Q}_p}A)[[t]]} D^{m,+}_{\mathrm{dif},A}$ for $m'\geq m$)

$$(M^r_A, D^{m,+}_{\mathrm{dif},A}, \alpha_m) \mapsto \{x\in M^r_A \mid \iota_{m'}(x)\in D^{m',+}_{\mathrm{dif},A} \subset D^{m',+}_{\mathrm{dif},A}[\frac{1}{t}] \overset{\alpha_m}{\simeq} D^{m'}_{\mathrm{dif}}(M^r_A), \forall m'\geq m\}.$$

Moreover, the equivalence commutes with arbitrary base change.

Proof. This follows from the Beauville-Laszlo lemma [BL95] and the consideration of φ-actions. A finite projective φ-module $D^r_{A,K}$ over $\mathcal{R}^r_A$ is equivalently the global section of a φ-bundle $\mathcal{D}^r_A$ over the relative annulus $\mathbb{U}^r\times_{\mathbb{Q}_p}\mathrm{Sp}(K'_0\otimes_{\mathbb{Q}_p} A)$ (see [KPX14, Prop. 2.2.7]). In terms of vector bundles, a φ-module M^r_A over $\mathcal{R}^r_{A,K}[\frac{1}{t}]$ corresponds to a φ-bundle over $\mathbb{U}^r\times_{\mathbb{Q}_p}\mathrm{Sp}(K'_0\otimes_{\mathbb{Q}_p} A)$ up to modifications along the divisors cut out by $Q_{m'}(X_{\mathbb{Q}_p})$ for $m'\geq m=m(r)$. Using the φ-action, modifications of D^r_A are determined by the modification at $Q_m(X_{\mathbb{Q}_p})$ which is recorded as the Γ_K-lattice $D^{m,+}_{\mathrm{dif}}(D^r_A)$ inside $D^m_{\mathrm{dif}}(D^r_A)$. See [Fru23, Thm. 5.11] for more details. □

A.2. **Almost de Rham families.** We prove [EGH23, Prop. 5.3.27]. We need to write the proof here because some constructions in the proof are used for the main theorem.

Let $K/\mathbb{Q}_p$ be a local field, A be an affinoid algebra over L. We fix an embedding $\tau: K\hookrightarrow L\subset A$. A Γ_K-representation $D^{m,+}_{\mathrm{dif},A}$ of rank n with coefficients in $A\otimes_K K_m$ is a finite projective $(A\otimes_K K_m)[[t]]$-module of rank n with a continuous semilinear action of Γ_K, where Γ_K acts on K_m by the Galois action and on t via the cyclotomic character. Write $D^{m,+}_{\mathrm{Sen},A} = D^{m,+}_{\mathrm{dif},A}/t$. As in [Fon04, Prop. 3.7], differentiate the Γ_K-action, we can obtain a connection ∇ on $D^{m,+}_{\mathrm{dif},A} = \varprojlim_k D^{m,+}_{\mathrm{dif},A}/t^k$ which is the Sen operator after modulo t. We say that $D^{m,+}_{\mathrm{dif},A}$ is almost de Rham of weights $\mathbf{h}=(h_1,\cdots,h_n)\in\mathbb{Z}^n, h_1\leq\cdots\leq h_n$ if all the Sen polynomial of the specialization $D^{m,+}_{\mathrm{Sen},x}$ is equal to $\prod_{i=1}^n (T-h_i)$ for any $x\in\mathrm{Sp}(A)$. Write $D^m_{\mathrm{dif},A} = D^{m,+}_{\mathrm{dif},A}[\frac{1}{t}]$. Denote by $\mathfrak{S}_{m,A} = (A\otimes_K K_m)[[t]]$. Let $\mathrm{Rep}_{\mathfrak{S}_{m,A}}(\Gamma_K)_{\mathbf{h}}$ be the groupoid of almost de Rham semilinear Γ_K-representations over projective $\mathfrak{S}_{m,A}$-modules of weight $\mathbf{h}$. For $m'\geq m$, the tensor product $-\otimes_{K_m} K_{m'}$ induces a functor $\mathrm{Rep}_{\mathfrak{S}_{m,A}}(\Gamma_K)_{\mathbf{h}} \to \mathrm{Rep}_{\mathfrak{S}_{m',A}}(\Gamma_K)_{\mathbf{h}}$.

Definition A.4. Write $\mathfrak{S}_{\infty,A} = (A\otimes_K K_\infty)[[t]]$.

(1) Define the groupoid $\mathrm{Rep}_{\mathfrak{S}_{\infty,A}}(\Gamma_K)_{\mathbf{h}} := \varinjlim_m \mathrm{Rep}_{\mathfrak{S}_{m,A}}(\Gamma_K)_{\mathbf{h}}$. Objects of $\mathrm{Rep}_{\mathfrak{S}_{\infty,A}}(\Gamma_K)_{\mathbf{h}}$ consist of representations $D^{m,+}_{\mathrm{dif},A}\in\mathrm{Rep}_{\mathfrak{S}_{m,A}}(\Gamma_K)_{\mathbf{h}}$ for some m identified with $D^{m,+}_{\mathrm{dif},A}\otimes_{K_m} K'_m \in \mathrm{Rep}_{\mathfrak{S}_{m',A}}(\Gamma_K)_{\mathbf{h}}$ for all $m'\geq m$. And for $D^{',m',+}_{\mathrm{dif},A}\in\mathrm{Rep}_{\mathfrak{S}_{m',A}}(\Gamma_K)_{\mathbf{h}}$,

$$\mathrm{Hom}_{\mathrm{Rep}_{\mathfrak{S}_{\infty,A}}(\Gamma_K)_{\mathbf{h}}}(D^{m,+}_{\mathrm{dif},A}, D^{',m',+}_{\mathrm{dif},A}) = \varinjlim_{m''\geq m,m'} \mathrm{Hom}_{\mathrm{Rep}_{\mathfrak{S}_{m'',A}}(\Gamma_K)_{\mathbf{h}}}(D^{m'',+}_{\mathrm{dif},A}, D^{',m'',+}_{\mathrm{dif},A})$$

where $D^{m'',+}_{\mathrm{dif},A} := D^{m,+}_{\mathrm{dif},A}\otimes_{K_m} K_{m''}, D^{',m'',+}_{\mathrm{dif},A} := D^{',m',+}_{\mathrm{dif},A}\otimes_{K_{m'}} K_{m''}$.

(2) For $D^{m,+}_{\mathrm{dif},A}\in\mathrm{Rep}_{\mathfrak{S}_{m,A}}(\Gamma_K)_{\mathbf{h}}$, we say that m is large enough if the action of Γ_{K_m} on $D^{m,+}_{\mathrm{dif},A}/t$ is analytic, namely for any $\gamma\in\Gamma_{K_m}$, the action of γ is given by the convergent series $\exp(\log(\epsilon(\gamma))\nabla) = \sum_{i=0}^\infty \frac{(\log(\epsilon(\gamma))\nabla)^i}{i!}$.

For any $D^{m,+}_{\mathrm{dif},A}\in\mathrm{Rep}_{\mathfrak{S}_{m,A}}(\Gamma_K)_{\mathbf{h}}$, there exists always $m'>m$ such that m'' is large enough for $D^{m'',+}_{\mathrm{dif},A}$ for any $m''\geq m'$ (such that the series $\log(\gamma_{K_{m'}})$ in $\mathrm{End}_{A\times_K K_{m'}}(D^{m',+}_{\mathrm{dif},A}/t)$ converges and $\gamma_{K_{m'}} = \exp(\log(\gamma_{K_{m'}}))$, cf. [KPX14, Prop. 2.2.14]).

We take $G=\mathrm{GL}_{n/L}$ and $P_{\mathbf{h}}$ a standard parabolic subgroup of G such that the Weyl group of the Levi of $P_{\mathbf{h}}$ is the stabilizer of $\mathbf{h}$ in $\mathcal{S}_n$ as in §3. Let $(\widetilde{\mathfrak{g}}_{\mathbf{h}}/G)^\wedge_0(A)$ be the groupoid of triples $(D_{\mathrm{pdR,A}}, \nu_A, \mathrm{Fil}^\bullet D_{\mathrm{pdR},A})$ where $D_{\mathrm{pdR,A}}$ is a finite projective A-module of rank n, $\mathrm{Fil}^\bullet D_{\mathrm{pdR},A}$ is a decreasing filtration of projective sub-A-modules of type $\mathbf{h}$ as in Definition 3.4 and $\nu_A\in\mathrm{End}_A(D_{\mathrm{pdR,A}})$ is a nilpotent endomorphism which keeps the filtration.

We define the functor $D_{\mathrm{pdR}}: \mathrm{Rep}_{\mathfrak{S}_{\infty,A}}(\Gamma_K)_{\mathbf{h}} \to (\widetilde{\mathfrak{g}}_{\mathbf{h}}/G)^\wedge_0(A)$.

Definition A.5. Let $\log(t)$ be the formal variable such that $\gamma.\log(t)=\log(\epsilon(\gamma))+\log(t)$ for $\gamma\in\Gamma_K$ and let the operator ν_A act on $\mathfrak{S}_{m,A}[\log(t)]$ as a $\mathfrak{S}_{m,A}$-linear derivative such that $\nu_A(\log(t))=-1$.

(1) For $D^{m,+}_{\mathrm{dif},A}\in\mathrm{Rep}_{\mathfrak{S}_{m,A}}(\Gamma_K)_{\mathbf{h}}$ such that m is large enough, define the A-module with a filtration

$$D_{\mathrm{pdR}}(D^{m,+}_{\mathrm{dif},A})=D_{\mathrm{pdR}}(D^{m}_{\mathrm{dif},A}):=(D^{m}_{\mathrm{dif},A}\otimes_{\mathfrak{S}_{m,A}}\mathfrak{S}_{m,A}[\log(t)])^{\Gamma_K},$$
$$\mathrm{Fil}^iD_{\mathrm{pdR}}(D^{m,+}_{\mathrm{dif},A}):=(t^iD^{m,+}_{\mathrm{dif},A}\otimes_{\mathfrak{S}_{m,A}}\mathfrak{S}_{m,A}[\log(t)])^{\Gamma_K}$$

for $i\in\mathbb{Z}$, equipped with an A-linear operator ν_A induced by the derivative ν_A on $\mathfrak{S}_{m,A}[\log(t)]$.

(2) Given $(D_{\mathrm{pdR,A}},\nu_A,\mathrm{Fil}^\bullet D_{\mathrm{pdR},A})\in(\widetilde{\mathfrak{g}}_{\mathbf{h}}/G)^\wedge_0(A)$, we let

$$D^m_{\mathrm{dif}}(D_{\mathrm{pdR,A}},\nu_A):=(D_{\mathrm{pdR},A}\otimes_A\mathfrak{S}_{m,A}[\frac{1}{t}][\log(t)])^{\nu_A=0}$$

and

$$D^{m,+}_{\mathrm{dif}}(D_{\mathrm{pdR,A}},\nu_A,\mathrm{Fil}^\bullet D_{\mathrm{pdR},A}):=(\sum_{i\in\mathbb{Z}}\mathrm{Fil}^iD_{\mathrm{pdR},A}\otimes_A t^{-i}\mathfrak{S}_{m,A}[\log(t)])^{\nu_A=0}$$

be $\mathfrak{S}_{m,A}$-modules equipped with actions of Γ_K.

Proposition A.6. *The functors $D_{\mathrm{pdR}}(-),D^{m,+}_{\mathrm{dif}}(-)$ are well defined and the following statements hold.*

(1) If m is large enough for $D^{m,+}_{\mathrm{dif},A}\in\mathrm{Rep}_{\mathfrak{S}_{m,A}}(\Gamma_K)_{\mathbf{h}}$ in the sense of Definition A.4, then the natural map

$$\rho_{\mathrm{pdR},m}:\sum_{i\in\mathbb{Z}}\mathrm{Fil}^iD_{\mathrm{pdR}}(D^{m,+}_{\mathrm{dif},A})\otimes_A t^{-i}\mathfrak{S}_{m,A}[\log(t)]\to D^{m,+}_{\mathrm{dif},A}\otimes_{\mathfrak{S}_{m,A}}\mathfrak{S}_{m,A}[\log(t)].$$

is an isomorphism and induces an isomorphism in $\mathrm{Rep}_{\mathfrak{S}_{m,A}}(\Gamma_K)_{\mathbf{h}}$:

$$D^{m,+}_{\mathrm{dif}}(D_{\mathrm{pdR}}(D^{m,+}_{\mathrm{dif},A}),\nu_A,\mathrm{Fil}^\bullet D_{\mathrm{pdR}}(D^{m,+}_{\mathrm{dif},A}))\simeq D^{m,+}_{\mathrm{dif},A}.$$

(2) The functors $D_{\mathrm{pdR}}(-)$ and $D^{m,+}_{\mathrm{dif}}(-)$ induce an equivalence of groupoids $\mathrm{Rep}_{\mathfrak{S}_{\infty,A}}(\Gamma_K)_{\mathbf{h}}\simeq(\widetilde{\mathfrak{g}}_{\mathbf{h}}/G)^\wedge_0(A)$. Moreover, the equivalence commutes with arbitrary base change.

Proof. Step 1. We show that the functor $D^{m,+}_{\mathrm{dif}}(-)$ is well-defined. Suppose that

$$(D_{\mathrm{pdR,A}},\nu_A,\mathrm{Fil}^\bullet D_{\mathrm{pdR},A})\in(\widetilde{\mathfrak{g}}_{\mathbf{h}}/G)^\wedge_0(A).$$

Note that $D^m_{\mathrm{dif}}(D_{\mathrm{pdR,A}},\nu_A)=(D_{\mathrm{pdR},A}\otimes_A\mathfrak{S}_{m,A}[\log(t)])^{\nu_A=0}[\frac{1}{t}]$. We show that $(D_{\mathrm{pdR},A}\otimes_A\mathfrak{S}_{m,A}[\log(t)])^{\nu_A=0}$ is an almost de Rham Γ_K-representation of weight 0. First,

$$\nu_A(\sum_{i\geq0}x_i\log(t)^i)=\sum_{i\geq0}(\nu_A(x_i)\log(t)^i-x_{i+1}(i+1)\log(t)^i)=0$$

for $x_i\in D_{\mathrm{pdR},A}\otimes_A\mathfrak{S}_{m,A}$ if and only if $x_{i+1}=\frac{1}{i+1}\nu_A(x_i)$ for $i\geq0$. Since ν_A is nilpotent on $D_{\mathrm{pdR}}\otimes_A\mathfrak{S}_{m,A}$, we have an identification

$$D_{\mathrm{pdR},A}\otimes_A\mathfrak{S}_{m,A}\simeq(D_{\mathrm{pdR},A}\otimes_A\mathfrak{S}_{m,A}[\log(t)])^{\nu_A=0}$$
$$x\mapsto\sum_{i\geq0}\frac{1}{i!}\nu_A^i(x)\log(t)^i.$$

as $\mathfrak{S}_{m,A}$-modules (but not Γ_K-equivariantly). Under this identification, the connection ∇ on $(D_{\mathrm{pdR},A}\otimes_A\mathfrak{S}_{m,A}[\log(t)])^{\nu_A=0}$ corresponds to $\nabla+\nu_A$ on $D_{\mathrm{pdR},A}\otimes_A\mathfrak{S}_{m,A}$: $\nabla(\sum_i x_i\log(t)^i)=\sum_i(\nabla(x_i)+(i+1)x_{i+1})\log(t)^i=\sum_i(\nabla(x_i)+\nu_A(x_i))\log(t)^i$ where ν_A is linear for $\mathfrak{S}_{m,A}$ and ∇ kills $D_{\mathrm{pdR},A}$. Since ν_A is nilpotent, the Sen weights are pointwisely all zero.

Same argument shows that

$$D^{m,+}_{\mathrm{dif}}(D_{\mathrm{pdR,A}},\nu_A,\mathrm{Fil}^\bullet D_{\mathrm{pdR},A})\simeq\sum_i\mathrm{Fil}^iD_{\mathrm{pdR},A}\otimes_A t^{-i}\mathfrak{S}_{m,A}$$

as an $\mathfrak{S}_{m,A}$-module. Since $\mathrm{Fil}^\bullet D_{\mathrm{pdR},A}$ has projective graded pieces, we may choose a splitting $D_{\mathrm{pdR},A}=D_1\oplus D_2\cdots\oplus D_s$ where $0\neq D_i=\mathrm{Fil}^{k_i}D_{\mathrm{pdR},A}/\mathrm{Fil}^{k_i+1}D_{\mathrm{pdR},A}$ and $k_1>\cdots>k_s$ such that $\{-k_1,\cdots,-k_s\}=\{h_1,\cdots,h_n\}$ as sets. Then

$$D^{m,+}_{\mathrm{dif}}(D_{\mathrm{pdR,A}},\nu_A,\mathrm{Fil}^\bullet D_{\mathrm{pdR},A})=D_1\otimes_A t^{-k_1}\mathfrak{S}_{m,A}\oplus\cdots\oplus D_s\otimes_A t^{-k_s}\mathfrak{S}_{m,A}$$

is projective of rank n over $\mathfrak{S}_{m,A}$ of weights $\mathbf{h}$. If $m'\geq m$, by the above description, we see

$$D_{\mathrm{dif}}^{m',+}(D_{\mathrm{pdR,A}},\nu_A,\mathrm{Fil}^\bullet D_{\mathrm{pdR},A})=D_{\mathrm{dif}}^{m,+}(D_{\mathrm{pdR,A}},\nu_A,\mathrm{Fil}^\bullet D_{\mathrm{pdR},A})\otimes_{K_m}K_{m'}.$$

Hence the image in $\mathrm{Rep}_{\mathfrak{S}_{\infty,A}}(\Gamma_K)_{\mathbf{h}}$ is independent of m.

Step 2. Take $D_{\mathrm{dif},A}^{m,+}\in\mathrm{Rep}_{\mathfrak{S}_{m,A}}(\Gamma_K)_{\mathbf{h}}\subset\mathrm{Rep}_{\mathfrak{S}_{\infty,A}}(\Gamma_K)_{\mathbf{h}}$ such that m is large enough. We show that $(D_{\mathrm{pdR}}(D_{\mathrm{dif},A}^{m,+}),\nu_A,\mathrm{Fil}^\bullet D_{\mathrm{pdR}}(D_{\mathrm{dif},A}^{m,+}))\in(\widetilde{\mathfrak{g}}_{\mathbf{h}}/G)_0^\wedge(A)$ and is independent of m.

Recall $D_{\mathrm{Sen},A}^m=D_{\mathrm{dif},A}^{m,+}/t$. Consider the Γ_K-representation $t^iD_{\mathrm{Sen},A}^m$. There is a canonical Γ_K-decomposition $D_{\mathrm{Sen},A}^m=\oplus_{i=1}^sD_{\mathrm{Sen},A}^m\{\nabla_{\mathrm{Sen}}=-k_i\}$ according to the generalized eigenvalues of ∇_{Sen} where each $D_{\mathrm{Sen},A}^m\{\nabla_{\mathrm{Sen}}=-k_i\}$ is projective over $A\otimes_KK_m$ of rank m_i.

By Lemma A.7, each $(t^iD_{\mathrm{Sen},A}^m\otimes_{K_m}K_m[\log(t)])^{\Gamma_K}$ is finite projective over A of rank the multiplicity of $-i$ in $\mathbf{h}$ and the map

$$\oplus_{i\in\mathbb{Z}}t^{-i}(t^iD_{\mathrm{Sen},A}^m\otimes_{K_m}K_m[\log(t)])^{\Gamma_K}\otimes_KK_m[\log(t)]\to D_{\mathrm{Sen},A}^m\otimes_{K_m}K_m[\log(t)] \tag{A.1}$$

is an isomorphism of Γ_K-representations over $(A\otimes_KK_m)[\log(t)]$. Set

$$\mathrm{gr}^iD_{\mathrm{pdR}}(D_{\mathrm{dif},A}^{m,+}):=(t^iD_{\mathrm{dif},A}^{m,+}\otimes_{\mathfrak{S}_{m,A}}\mathfrak{S}_{m,A}[\log(t)])^{\Gamma_K}/(t^{i+1}D_{\mathrm{dif},A}^{m,+}\otimes_{\mathfrak{S}_{m,A}}\mathfrak{S}_{m,A}[\log(t)])^{\Gamma_K}.$$

There is an injection for all i

$$\mathrm{gr}^iD_{\mathrm{pdR}}(D_{\mathrm{dif},A}^{m,+})\hookrightarrow(t^iD_{\mathrm{Sen},A}^m\otimes_{K_m}K_m[\log(t)])^{\Gamma_K}$$

which we claim is an isomorphism.

Suppose that $v\in t^iD_{\mathrm{dif},A}^{m,+}$ with image $\overline{v}\in t^iD_{\mathrm{Sen},A}^m$ such that

$$\sum_{i=0}^\infty\frac{(-1)^i}{i!}\nabla^i(\overline{v})\log(t)^i\in(t^iD_{\mathrm{Sen},A}^m\otimes_{K_m}K_m[\log(t)])^{\nabla=0}.$$

Take $a\leq b\in\mathbb{Z}$ such that $i,h_1,\cdots,h_n\in[a,b]$. In the beginning of the proof of [Wu21, Prop. A.10], there exists $l\geq1$ and maps $\beta_k:t^{-b}D_{\mathrm{dif},A}^{m,+}\to t^{-b}D_{\mathrm{dif},A}^{m,+},k\geq1$ such that $\beta_k(v)-\beta_{k+1}(v)\in t^{k+1-a}D_{\mathrm{dif},A}^{m,+}$ and $(\gamma_{K_m}-1)^l\beta_k(x)\in t^{k+1-b}D_{\mathrm{dif},A}^{m,+}$. Moreover, l is large enough such that $(\gamma_{K_m}-1)^l(t^iD_{\mathrm{dif},A}^{m,+}/t^{i+1}D_{\mathrm{dif},A}^{m,+})^{(\gamma_{K_m}-1)-\mathrm{nil}}=0$. By the construction, β_k maps $t^iD_{\mathrm{dif},A}^{m,+}$ to $t^iD_{\mathrm{dif},A}^{m,+}$ and induces an automorphism of $(t^iD_{\mathrm{dif},A}^{m,+}/t^{i+1}D_{\mathrm{dif},A}^{m,+})^{(\gamma_{K_m}-1)-\mathrm{nil}}$ independent of k. Take any v' such that $\beta_k(v')$ has image $\overline{v}$ in $t^iD_{\mathrm{Sen},A}^m$. Then $\tilde{v}:=\varinjlim_k\beta_k(v')$ is a lift of $\overline{v}$ in $(t^iD_{\mathrm{dif},A}^{m,+})^{(\gamma_{K_m}-1)-\mathrm{nil}}$. We get that ∇ acts nilpotently on $\tilde{v}$ and $\sum_{i=0}^\infty\frac{(-1)^i}{i!}\nabla^i(\tilde{v})\log(t)^i\in(t^iD_{\mathrm{dif},A}^{m,+}\otimes_{K_m}K_m[\log(t)])^{\Gamma_{K_m}}$ (cf. the proof of [Wu21, Lem. A.2]). We conclude that the map $(t^iD_{\mathrm{dif},A}^{m,+}\otimes_{\mathfrak{S}_{m,A}}\mathfrak{S}_{m,A}[\log(t)])^{\Gamma_{K_m}}\to(t^iD_{\mathrm{Sen},A}^m\otimes_{K_m}K_m[\log(t)])^{\Gamma_{K_m}}$ is surjective. Take Γ_K-invariants we see

$$\mathrm{gr}^iD_{\mathrm{pdR}}(D_{\mathrm{dif},A}^{m,+})\simeq(t^iD_{\mathrm{Sen},A}^m\otimes_{K_m}K_m[\log(t)])^{\Gamma_K}.$$

Thus $D_{\mathrm{pdR}}(D_{\mathrm{dif},A}^{m,+})$ equipped with the filtration $\mathrm{Fil}^iD_{\mathrm{pdR}}(D_{\mathrm{dif},A}^{m,+})=(t^iD_{\mathrm{dif},A}^{m,+}\otimes_{\mathfrak{S}_{m,A}}\mathfrak{S}_{m,A}[\log(t)])^{\Gamma_K}$ has type $\mathbf{h}$ (we use that extensions of projective modules are still projective). And if $m'\geq m$, $\mathrm{gr}^iD_{\mathrm{pdR}}(D_{\mathrm{dif},A}^{m,+})=\mathrm{gr}^iD_{\mathrm{pdR}}(D_{\mathrm{dif},A}^{m',+})$ for all i by Lemma A.7. Hence the functor $D_{\mathrm{pdR}}(-)$ is independent of large enough m.

Step 3. We show that the functors induce an equivalence of categories. We first show that the map $\rho_{\mathrm{pdR},m}$ in (1) is an isomorphism. By Step 1, both sides are finite projective over $\mathfrak{S}_{m,A}[\log(t)]$. We may choose a splitting of the filtration $\mathrm{Fil}^\bullet D_{\mathrm{pdR}}(D_{\mathrm{dif},A}^{m,+})$ with graded pieces identified with $(t^iD_{\mathrm{Sen},A}^m\otimes_{K_m}K_m[\log(t)])^{\Gamma_K}$. Modulo t, $\rho_{\mathrm{pdR},m}$ coincides with the isomorphism (A.1), which is an injection. We get that $\rho_{\mathrm{pdR},m}$ is an injection itself (since the source is t-adically separated). To see the surjectivity, we only need to show that $\nu=0$ part of the lefthand side $D_{\mathrm{dif},A}^{m,+}$ is contained in the image. The $\nu=0$ part of the righthand side, which is a projective $\mathfrak{S}_{m,A}$-module, admits an explicit description in terms of $\mathrm{gr}^\bullet D_{\mathrm{pdR}}(D_{\mathrm{dif},A}^{m,+})$ by Step 1. Modulo t, the map between $\nu=0$ part is given by the $\nu=0$ part of (A.1), the surjectivity follows. This also shows that the map $D_{\mathrm{pdR}}(-)$ is essentially surjective. By the definition of the groupoid $\mathrm{Rep}_{\mathfrak{S}_{\infty,A}}(\Gamma_K)_{\mathbf{h}}$, $\mathrm{Hom}(D_{\mathrm{dif},A}^{m,+},D_{\mathrm{dif},A}^{m',+})=\varinjlim_{m''}\mathrm{Hom}(D_{\mathrm{dif},A}^{m,+}\otimes_{K_m}K_{m''},D_{\mathrm{dif},A}^{m',+}\otimes_{K_{m'}}K_{m''})$. To show fully faithfulness, we only need to show that for m large enough, the map $\mathrm{End}_{\mathrm{Rep}_{\mathfrak{S}_{m,A}}(\Gamma_K)_{\mathbf{h}}}(D_{\mathrm{dif},A}^{m,+})\to\mathrm{End}_{(\widetilde{\mathfrak{g}}_{\mathbf{h}}/G)_0^\wedge(A)}(D_{\mathrm{pdR}}(D_{\mathrm{dif},A}^{m,+}))$ is an isomorphism. The map is an injection by taking $\nu=0$ part of the canonical isomorphism $\rho_{\mathrm{pdR},m}$. The surjectivity follows similarly using $\rho_{\mathrm{pdR},m}$.

Step 4. Assume that $\mathrm{Sp}(B)\to\mathrm{Sp}(A)$ is a morphism of affinoids. We need to show that for m large enough, the natural map $\mathrm{Fil}^iD_{\mathrm{pdR}}(D^{m,+}_{\mathrm{dif},A})\otimes_A B\to\mathrm{Fil}^iD_{\mathrm{pdR}}(D^{m,+}_{\mathrm{dif},A}\otimes_{\mathfrak{S}_{m,A}}\mathfrak{S}_{m,B})$ is an isomorphism. Both sides are finite projective B-modules of the same rank, we can reduce to show that $\mathrm{gr}^iD_{\mathrm{pdR}}(D^{m,+}_{\mathrm{dif},A})\otimes_A B\to\mathrm{gr}^iD_{\mathrm{pdR}}(D^{m,+}_{\mathrm{dif},A}\otimes_{\mathfrak{S}_{m,A}}\mathfrak{S}_{m,B})$ is an isomorphism for all i. This follows from the proof of Lemma A.7 (see also [Bel15, §3]). □

Lemma A.7. *Let $D^m_{\mathrm{Sen},0}$ be a semilinear Γ_K-representation over a finite projective $A\otimes_K K_m$-module of rank n and with Sen weights pointwisely 0 such that m is large enough as in Definition A.4. Then for $i\neq 0$, we have $(t^iD^m_{\mathrm{Sen},0}\otimes_{K_m}K_\infty[\log(t)])^{\Gamma_K}=0$, and the map*

$$(D^m_{\mathrm{Sen},0}\otimes_{K_m}K_m[\log(t)])^{\Gamma_K}\otimes_K K_m[\log(t)]\to D^m_{\mathrm{Sen},0}\otimes_{K_m}K_m[\log(t)]$$

is an isomorphism of Γ_K-representations. Moreover, for $m'\geq m$, $(D^m_{\mathrm{Sen},0}\otimes_{K_m}K_m[\log(t)])^{\Gamma_K}=(D^m_{\mathrm{Sen},0}\otimes_{K_m}K_{m'}[\log(t)])^{\Gamma_K}$.

Proof. Weights zero means that the Sen operator ∇ acts nilpotently on $D^m_{\mathrm{Sen},0}$ by Lemma 3.9. Since m is large enough, we have $D^m_{\mathrm{Sen},0}\simeq(D^m_{\mathrm{Sen},0}\otimes_{K_m}K_m[\log(t)])^{\Gamma_{K_m}}=(D^m_{\mathrm{Sen},0}\otimes_{K_m}K_m[\log(t)])^{\nabla=0}$, cf. [Wu21, Lem. A.2]. Actually, we have $D^m_{\mathrm{Sen},0}=D^m_{\mathrm{Sen},0}\{\gamma_{K_m}=1\}$ and an identification of $A\otimes_K K_m$-modules

$$F:D^m_{\mathrm{Sen},0}\simeq(D^m_{\mathrm{Sen},0}\otimes_{K_m}K_m[\log(t)])^{\Gamma_{K_m}}:x\mapsto\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\nabla^i(x)\log(t)^i.$$

We show that the natural Γ_K-map

$$\rho_{\mathrm{Sen},m}:(D^m_{\mathrm{Sen},0}\otimes_{K_m}K_m[\log(t)])^{\Gamma_{K_m}}\otimes_{K_m}K_m[\log(t)]\to D^m_{\mathrm{Sen},0}\otimes_{K_m}K_m[\log(t)]$$

is an isomorphism. Since $F:\nabla(x)\mapsto\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\nabla^{i+1}(x)\log(t)^i$ and ∇ is nilpotent, one can verify that the map $\rho_{\mathrm{Sen},m}$ is a surjection by a decreasing induction: for any $x\in D^m_{\mathrm{Sen},0}$ and i, we have $\nabla^j(x)\otimes 1$ is in the image of $\rho_{\mathrm{Sen},m}$ for all $j\geq i$. To show the injectivity, consider the $A\otimes_K K_m$-linear derivation of ν on the two sides: $\nu(\log(t))=-1$. Under the identification F, ν corresponds to ∇ on $D^m_{\mathrm{Sen},0}$. The map $\rho_{\mathrm{Sen},m}$ induces an isomorphism on $\nu=0$ part: $\nu(\sum_i F(x_i)\log(t)^i)=0$ if and only if $(i+1)x_{i+1}=\nabla(x_i)$ and in this case we have

$$\rho_{\mathrm{Sen},m}(\sum_i F(x_i)\log(t)^i)=\sum_{i+j=k}\frac{(-1)^i}{i!j!}\nabla^k(x_0)\log(t)^k\neq 0$$

if $x_0\neq 0$. Now suppose that $\rho_{\mathrm{Sen},m}(x)=0$ for some $x=\sum_i F(x_i)\log(t)^i\neq 0$. Let i be the minimal integer such that $\nu^i(x)=0$. Then $i\geq 1$ and $\rho_{\mathrm{Sen},m}(\nu^{i-1}(x))=0$ which forces $\nu^{i-1}(x)=0$, contradiction! Hence $\rho_{\mathrm{Sen},m}$ is an isomorphism.

Finally, by Galois descent, we have $(D^m_{\mathrm{Sen},0}\otimes_{K_m}K_m[\log(t)])^{\Gamma_{K_m}}=(D^m_{\mathrm{Sen},0}\otimes_{K_m}K_m[\log(t)])^{\Gamma_K}\otimes_K K_m$ as subspaces of $D^m_{\mathrm{Sen},0}\otimes_{K_m}K_m[\log(t)]$. And for $m'\geq m$, we have an isomorphism

$$(D^m_{\mathrm{Sen},0}\otimes_{K_m}K_m[\log(t)])^{\Gamma_K}\otimes_K K_{m'}[\log(t)]\to D^m_{\mathrm{Sen},0}\otimes_{K_m}K_{m'}[\log(t)].$$

Taking $\Gamma_{K_{m'}}$-invariants, we see $(D^m_{\mathrm{Sen},0}\otimes_{K_m}K_{m'}[\log(t)])^{\Gamma_{K_{m'}}}=(D^m_{\mathrm{Sen},0}\otimes_{K_m}K_{m'}[\log(t)])^{\nabla=0}=(D^m_{\mathrm{Sen},0}\otimes_{K_m}K_m[\log(t)])^{\Gamma_K}\otimes_K K_{m'}$. Taking Γ_K-invariants we get $(D^m_{\mathrm{Sen},0}\otimes_{K_m}K_{m'}[\log(t)])^{\Gamma_K}=(D^m_{\mathrm{Sen},0}\otimes_{K_m}K_m[\log(t)])^{\Gamma_K}$. □

Appendix B. GAGA and formal functions

We consider formal completions of rigid spaces and coherent modules on these spaces.

Definition B.1. Let $\mathfrak{X}$ be a rigid space over L and let $\mathcal{I}$ be a coherent sheaf of ideals of $\mathcal{O}_{\mathfrak{X}}$. Let $\mathfrak{X}_n$ be the analytic closed subspace of $\mathfrak{X}$ defined by $\mathcal{I}^n$. The formal completion of $\mathfrak{X}$ along $\mathfrak{X}_1$, denoted by $\mathfrak{X}^\wedge$, is the ringed site $(\mathfrak{X}_1,\mathcal{O}_{\mathfrak{X}^\wedge})$ which has the same underlying Grothendieck topological space as $\mathfrak{X}_1$ and the structure sheaf $\mathcal{O}_{\mathfrak{X}^\wedge}:=\varprojlim_n\mathcal{O}_{\mathfrak{X}_n}$.

The space above should be considered as a formal rigid analytic space, except that we will ignore the topology on the sheaf of the topological rings $\mathcal{O}_{\mathfrak{X}^\wedge}$.

We consider affinoid cases first. In the following, let A be an affinoid algebra over a p-adic field L and let $I\subset A$ be an ideal. Let $\mathfrak{Y}^\wedge=\varinjlim_n\mathfrak{Y}_n=\varinjlim_n\mathrm{Sp}(A/I^n)$ be the formal completion of $\mathfrak{Y}=\mathrm{Sp}(A)$ along $\mathrm{Sp}(A/I)$. For an affinoid open subspace $\mathrm{Sp}(B)\subset\mathrm{Sp}(A)$, $\mathcal{O}_{\mathfrak{Y}_n}(\mathrm{Sp}(B/I))=B\widehat{\otimes}_A A/I^n=B/I^n$ and $\mathcal{O}_{\mathfrak{Y}^\wedge}(B/I)=B^\wedge:=\varprojlim_n B/I^n$.

Lemma B.2. *An admissible open subset of* $\mathrm{Sp}(A/I)$ *admits a covering by open affinoids of the form* $\mathrm{Sp}(B/I)$ *for affinoid opens* $\mathrm{Sp}(B) \subset \mathrm{Sp}(A)$.

Proof. A rational subdomain of $\mathrm{Sp}(A/I)$ has the form $\mathrm{Sp}((A/I)\langle x_1, \cdots, x_n\rangle/(x_1 g - f_1, \cdots, x_n g - f_n))$ for some $f_i, g \in A/I$ such that f_i, g generate the unit ideal of A/I. Take lifts $\widetilde{f_i}, \widetilde{g}$ in A and add possibly some $\widetilde{f}_{n+1}, \cdots, \widetilde{f}_{n+k} \in I$, we see it has the form $\mathrm{Sp}(B/I)$ for a rational subdomain $\mathrm{Sp}(B) \subset \mathrm{Sp}(A)$ where $B = A\langle x_1, \cdots, x_{n+k}\rangle/(x_1\widetilde{g} - \widetilde{f}_1, \cdots, x_{n+k}\widetilde{g} - \widetilde{f}_{n+k})$. Then the statement follows from that rational subdomains form a basis for the Grothendieck topology [Bos14, Cor. 4.2/12]. □

Definition B.3. An $\mathcal{O}_{\mathfrak{Y}^\wedge}$-module $\mathcal{F}$ on the I-adic formal affinoid space $\mathfrak{Y}^\wedge$ is coherent if $\mathcal{F}$ has the form $\mathcal{F} = \varprojlim \mathcal{F}_n$ where $\mathcal{F}_n$ are coherent $\mathcal{O}_{\mathfrak{Y}_n}$-modules such that $\mathcal{F}_n/I^{n-1} = \mathcal{F}_{n-1}$.

Suppose that F is a finitely generated $A^\wedge$-module (a coherent $\mathcal{O}_{\mathrm{Spec}(A^\wedge)}$-module), we can associate a coherent $\mathcal{O}_{\mathfrak{Y}^\wedge}$-module $\mathcal{F} = \varprojlim \mathcal{F}_n$ where $\mathcal{F}_n$ is the coherent $\mathcal{O}_{\mathfrak{Y}_n}$-module attached to F/I^n. The following lemma is an analogue of [Gro60, Prop. 10.10.5, Ch.I].

Lemma B.4. *Let* $\mathfrak{Y}^\wedge$ *be as above.*

(1) Let $\mathcal{F} = \varprojlim_n \mathcal{F}_n$ *be a coherent* $\mathcal{O}_{\mathfrak{Y}^\wedge}$*-module. Then we have* $\mathcal{F}(\mathfrak{Y}^\wedge) = \varprojlim_n \mathcal{F}_n(\mathfrak{Y}_n)$ *and* $R^i\Gamma(\mathfrak{Y}^\wedge, \mathcal{F}) = 0$ *for* $i > 0$*. Moreover,* $\mathcal{F}_n$ *is uniquely determined by* $F := \mathcal{F}(\mathfrak{Y}^\wedge)$ *which is a finitely generated* $A^\wedge$*-module and* $\mathcal{F}_n(\mathfrak{Y}^\wedge) = F/I^n$*. And for any affinoid open* $\mathrm{Sp}(B) \subset \mathrm{Sp}(A)$ *which defines an affinoid open* $\mathrm{Sp}(B/I) \subset \mathrm{Sp}(A/I)$*, we have* $\mathcal{F}(\mathrm{Sp}(B/I)) = F \otimes_{A^\wedge} B^\wedge$.

(2) The functor $F \mapsto \mathcal{F}$ *from the category of finitely generated* $\mathcal{O}(\mathfrak{Y}^\wedge)$*-modules to the category of coherent* $\mathcal{O}_{\mathfrak{Y}^\wedge}$*-modules induces an equivalence of abelian categories.*

(3) An $\mathcal{O}_{\mathfrak{Y}^\wedge}$*-module* $\mathcal{F}$ *is coherent if and only if it is coherent in the sense of* [Sta24, Tag 03DK].

Proof. (1) Since the maps $\mathcal{F}_n \to \mathcal{F}_{n-1}$ are surjective, we have $\mathcal{F} = R\varprojlim_n \mathcal{F}_n$. Since $\mathfrak{Y}^\wedge$ is affinoid, the maps $\mathcal{F}_n(\mathfrak{Y}^\wedge) \to \mathcal{F}_{n-1}(\mathfrak{Y}^\wedge)$ are surjective. Hence $R\Gamma(\mathfrak{Y}^\wedge, \mathcal{F}) = \varprojlim_n \mathcal{F}_n(\mathfrak{Y}_n)$ by [Sta24, Tag 0D60]. By [Sta24, Tag 09B8], we have $\mathcal{F}(\mathfrak{Y}^\wedge)/I^n = \mathcal{F}_n(\mathfrak{Y}^\wedge)$ and $\mathcal{F}(\mathfrak{Y}^\wedge)$ is finitely generated by Nakayama lemma, see [Sta24, Tag 087W]. Finally, since $\mathcal{F}_n$ is coherent over $\mathfrak{Y}_n$ and $\mathrm{Sp}(B/I^n)$ is an affinoid open in $\mathrm{Sp}(A/I^n)$, $\mathcal{F}_n(\mathrm{Sp}(B/I^n)) = \mathcal{F}_n(A/I^n) \otimes_{A/I^n} B/I^n = F/I^n \otimes_{A/I^n} B/I^n$. Taking inverse limit $\mathcal{F}(\mathrm{Sp}(B/I^n)) = \varprojlim_n F \otimes_{A^\wedge} B/I^n = F \otimes_{A^\wedge} B^\wedge$ using that F is finitely generated over $A^\wedge$.

(2) The essential surjectivity is by (1). We need verify fully faithfulness. Let F, G be two finitely generated $A^\wedge$-modules and let $\mathcal{F}, \mathcal{G}$ be the corresponding coherent sheaves. Then $\mathrm{Hom}_{A^\wedge}(F, G) = \varprojlim_n \mathrm{Hom}_{A/I^n}(F/I^n, G/I^n) = \varprojlim_n \mathrm{Hom}_{\mathcal{O}_{\mathfrak{Y}^\wedge}}(\mathcal{F}_n, \mathcal{G}_n)$ by [Sta24, Tag 0EHN]. Apply this argument for the formal affinoid subspace $\mathrm{Sp}(B/I)$ of $\mathfrak{Y}^\wedge$, we get

$$\mathcal{H}om_{\mathcal{O}_{\mathfrak{Y}^\wedge}}(\mathcal{F}, \mathcal{G})(\mathrm{Sp}(B/I)) = \mathrm{Hom}_{B^\wedge}(\mathcal{F}(B^\wedge), \mathcal{G}(B^\wedge)) = \varprojlim_n \mathcal{H}om_{\mathcal{O}_{\mathfrak{Y}^\wedge}}(\mathcal{F}_n, \mathcal{G}_n)(\mathrm{Sp}(B/I)).$$

Hence $\mathcal{H}om_{\mathcal{O}_{\mathfrak{Y}^\wedge}}(\mathcal{F}, \mathcal{G}) = \varprojlim_n \mathcal{H}om_{\mathcal{O}_{\mathfrak{Y}^\wedge}}(\mathcal{F}_n, \mathcal{G}_n)$. Taking global sections we see $\mathrm{Hom}_{\mathcal{O}_{\mathfrak{Y}^\wedge}}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{A^\wedge}(F, G)$. This proves the equivalence. See [Sta24, Tag 087X] for the structure of abelian categories.

(3) The proof is the same as for [Gro60, Prop. 10.10.5, Ch.I], using that affinoid algebras as well as their completions are Noetherian. □

Let B be an affinoid algebra over L. Write $S = \mathrm{Spec}(B)$ and $\mathfrak{S} = \mathrm{Sp}(B)$. Suppose that $f : X_S \to \mathrm{Spec}(B)$ is a projective scheme over S and let $X_{\mathfrak{S}}$ be its relative analytification over $\mathfrak{S}$ which is equipped with a map of locally ringed spaces $\iota : X_{\mathfrak{S}} \to X_S$ (see [Con06, Exa. 2.2.11, Exa. 2.3.11]). The following result was firstly proved in [Köp74]. See also [Con06, Exa. 3.2.6] or [Poi10, Ann. A].

Theorem B.5 (Relative GAGA theorem)**.** *In the above situation, the functor* $\mathcal{F} \mapsto \iota^*\mathcal{F}$ *induces an equivalence of categories of coherent* $\mathcal{O}_{X_S}$*-modules and coherent* $\mathcal{O}_{X_{\mathfrak{S}}}$*-modules. And for any coherent* $\mathcal{O}_{X_S}$*-module* $\mathcal{F}$ *on* X_S*, the natural morphism* $H^i(X_S, \mathcal{F}) \to H^i(X_{\mathfrak{S}}, \iota^*\mathcal{F})$ *of finite* B*-modules is an isomorphism for all* $i \geq 0$.

We go to the formal setting. Suppose that $f_Y : X \to Y$ is a projective scheme over $Y = \mathrm{Spec}(A)$. Let $X_{Y_n} = X \times_Y Y_n$ where $Y_n = \mathrm{Spec}(A/I^n)$. We form relative analytification $X_{\mathfrak{Y}_n}$ for X_{Y_n} over

$\mathrm{Spec}(A/I^n)$ with proper maps $X_{\mathfrak{Y}_n} \to \mathfrak{Y}_n$ as well as $X_{\mathfrak{Y}} \to \mathfrak{Y}$. Then $X_{\mathfrak{Y}^\wedge} = \varinjlim_n X_{\mathfrak{Y}_n}$ is the formal completion of $X_{\mathfrak{Y}}$ along the closed subspace $X_{\mathfrak{Y}_1}$. We can define the category of coherent $\mathcal{O}_{X_{\mathfrak{Y}^\wedge}}$-modules as in Definition B.3, which is equivalent to the usual definition on a ringed site by Lemma B.4 (cf. [Gro60, Thm. 10.11.3, Ch.I]).

We also have a formal scheme $X_{\mathrm{Spf}(A^\wedge)} = \varinjlim_n X_{Y_n}$ and a scheme $X_{\mathrm{Spec}(A^\wedge)}$ the base change of $X_Y = X$ to $\mathrm{Spec}(A^\wedge)$. There are natural morphisms of ringed sites $X_{\mathfrak{Y}^\wedge} \to X_{\mathrm{Spf}(A^\wedge)} \to X_{\mathrm{Spec}(A^\wedge)}$. Write $f_{\mathfrak{Y}_n} : X_{\mathfrak{Y}_n} \to \mathfrak{Y}_n$. The following corollary generalizes Lemma B.4.

Corollary B.6. *Let $X_{\mathfrak{Y}^\wedge}, X_{\mathrm{Spf}(A^\wedge)}, X_{\mathrm{Spec}(A^\wedge)}$ be as above.*

(1) The category of coherent $\mathcal{O}_{X_{\mathfrak{Y}^\wedge}}$-modules is equivalent to the category of coherent modules on the formal scheme $X_{\mathrm{Spf}(A^\wedge)}$ (in the sense of [Sta24, Tag 089N]*). Both categories are equivalent to the category of coherent $\mathcal{O}_{X_{\mathrm{Spec}(A^\wedge)}}$-modules on the scheme $X_{\mathrm{Spec}(A^\wedge)}$.*

(2) Let $\mathcal{F} = \varprojlim_n \mathcal{F}_n$ be a coherent module on $X_{\mathfrak{Y}^\wedge}$ and $\mathcal{F}_{A^\wedge}$ be the corresponding coherent module on $X_{\mathrm{Spec}(A^\wedge)}$ by (1). Then for all $i \geq 0$, $R^i f_{\mathfrak{Y}^\wedge,*}\mathcal{F} = \varprojlim_n R^i f_{\mathfrak{Y}_n,*}\mathcal{F}_n$ and is the coherent $\mathcal{O}_{\mathfrak{Y}^\wedge}$-module attached to the finite $A^\wedge$-module $H^i(X_{\mathrm{Spec}(A^\wedge)}, \mathcal{F}_{A^\wedge})$.

Proof. (1) The first equivalence is an application of Theorem B.5 above for each $X_{\mathfrak{Y}_n}$ and by the definition of the category of coherent modules. The equivalence of coherent modules on $X_{\mathrm{Spf}(A^\wedge)}$ and $X_{\mathrm{Spec}(A^\wedge)}$ is Grothendieck's existence theorem [Sta24, Tag 08BE].

(2) Since $Rf_{\mathfrak{Y}^\wedge,*}\mathcal{F} = Rf_{\mathfrak{Y}^\wedge,*}R\varprojlim_n \mathcal{F}_n = R\varprojlim_n Rf_{\mathfrak{Y}^\wedge,*}\mathcal{F}_n$, we have $R^i f_{\mathfrak{Y}^\wedge,*}\mathcal{F} = \varprojlim_n R^i f_{\mathfrak{Y}_n,*}\mathcal{F}_n$ by [Sta24, Tag 0D60] for $i \geq 0$ provided that $R^1\varprojlim_n R^i f_{\mathfrak{Y}_n,*}\mathcal{F}_n = 0$ for all i. We need to show that the inverse system $R^i f_{\mathfrak{Y}_n,*}\mathcal{F}_n$ is Mittag-Leffler (cf. [Emm96]). Write $(\mathcal{F}_{A/I^n})_n$ for the coherent modules over $X_{\mathrm{Spf}(A^\wedge)}$ by the equivalence in (1). Then $(\mathcal{F}_{A/I^n})_n = (\mathcal{F}_{A^\wedge}/I^n)_n$. Under the equivalence in Theorem B.5, $R^i f_{\mathfrak{Y}_n,*}\mathcal{F}_n = R^i f_{Y_n,*}\mathcal{F}_{A/I^n}$ as coherent sheaves associated to the same A/I^n-module $H^i(X_{Y_n}, \mathcal{F}_{A/I^n}) = H^i(X_{\mathrm{Spec}(A^\wedge)}, \mathcal{F}_{A^\wedge}/I^n)$. Apply [Sta24, Tag 02OB] for the proper morphism $X_{\mathrm{Spec}(A^\wedge)} \to \mathrm{Spec}(A^\wedge)$ and the sheaf $\mathcal{F}_{A^\wedge}$, we see the system $(R^i f_{\mathfrak{Y}_n,*}\mathcal{F}_n)_n$ is Mittag-Leffler. By [Sta24, Tag 087U], $H^i(X_{\mathrm{Spec}(A^\wedge)}, \mathcal{F}_{A^\wedge}) = \varprojlim_n H^i(X_{\mathfrak{Y}_n}, \mathcal{F}_n)$ as finite $A^\wedge$-modules.

Hence $R^i f_{\mathfrak{Y}^\wedge,*}\mathcal{F}(\mathfrak{Y}^\wedge) = \varprojlim_n R^i f_{\mathfrak{Y}_n,*}\mathcal{F}_n(\mathfrak{Y}_n) = H^i(X_{\mathrm{Spec}(A^\wedge)}, \mathcal{F}_{A^\wedge})$ as $A^\wedge$-modules. For an affinoid subdomain $\mathrm{Sp}(B) \subset \mathrm{Sp}(A)$, a similar statement holds replacing $A^\wedge$ by $B^\wedge$. The ring map $A \to B$ is flat [Bos14, Cor. 4.1/5]. Hence the maps $A/I^n \to B/I^n$ and $A^\wedge \to B^\wedge$ are flat by Lemma B.7 below. By the flat base change, we have $H^i(X_{\mathrm{Spec}(B^\wedge)}, \mathcal{F}_{B^\wedge}) = H^i(X_{\mathrm{Spec}(A^\wedge)}, \mathcal{F}_{A^\wedge}) \otimes_{A^\wedge} B^\wedge$. Thus $R^i f_{\mathfrak{Y}^\wedge,*}\mathcal{F} = \varprojlim_n R^i f_{\mathfrak{Y}_n,*}\mathcal{F}_n$, as an $\mathcal{O}_{\mathfrak{Y}^\wedge}$-module, is the coherent $\mathcal{O}_{\mathfrak{Y}^\wedge}$-module attached to the finite $A^\wedge$-module $H^i(X_{\mathrm{Spec}(A^\wedge)}, \mathcal{F}_{A^\wedge})$ (see Lemma B.4). □

We used frequently the following lemma.

Lemma B.7. *Let A be a ring and let I be an ideal of A. Suppose that $(M_n)_n$ is an inverse system of A-modules such that M_n is a flat A/I^n-module for any n. Let $M := \varprojlim_n M_n$.*

(1) Suppose that A is Noetherian and that the transition maps $M_{n+1} \to M_n$ are surjective for all n, then M is flat over A and $Q \otimes_A M = \varprojlim_n Q \otimes_A M_n$ for any finite A-module Q.

(2) Suppose that $M_{n+1} \otimes_{A/I^{n+1}} A/I^n = M_n$ and M_n is finite flat over A/I^n for all n. If M_1 is finite projective over A/I, then M is finite projective over $A^\wedge = \varprojlim A/I^n$ and $M \otimes_{A^\wedge} A/I^n = M_n$ for all n.

Proof. (1) is [Sta24, Tag 0912]. (2) follows from [Sta24, Tag 0D4B]. □

References

[ABV92] Jeffrey Adams, Dan Barbasch, and David A. Vogan, Jr. *The Langlands classification and irreducible characters for real reductive groups*, volume 104 of *Prog. Math.* Boston, MA etc.: Birkhäuser, 1992.

[Bel15] Rebecca Bellovin. p-adic Hodge theory in rigid analytic families. *Algebra & Number Theory*, 9(2):371–433, 2015.

[Ber02] Laurent Berger. Représentations p-adiques et équations différentielles. *Inventiones mathematicae*, 148:219–284, 2002.

[Ber08a] Laurent Berger. Construction de (φ, Γ)-modules: représentations p-adiques et B-paires. *Algebra & Number Theory*, 2(1):91–120, 2008.

[Ber08b] Laurent Berger. Équations différentielles p-adiques et (φ, N)-modules filtrés. *Astérisque*, 319:13–38, 2008.

[BG80] Joseph N Bernstein and Sergei I Gelfand. Tensor products of finite and infinite dimensional representations of semisimple lie algebras. *Compositio Mathematica*, 41(2):245–285, 1980.

[BG99] Alexander Beilinson and Victor Ginzburg. Wall-crossing functors and $\mathcal{D}$-modules. *Representation Theory of the American Mathematical Society*, 3(1):1–31, 1999.

[BHS19] Christophe Breuil, Eugen Hellmann, and Benjamin Schraen. A local model for the trianguline variety and applications. *Publications mathématiques de l'IHÉS*, 130(1):299–412, 2019.

[BK07] Michel Brion and Shrawan Kumar. *Frobenius splitting methods in geometry and representation theory*, volume 231. Springer Science & Business Media, 2007.

[BK15] Erik Backelin and Kobi Kremnitzer. Singular localization of $\mathfrak{g}$-modules and applications to representation theory. *Journal of the European Mathematical Society*, 17(11):2763–2787, 2015.

[BL95] Arnaud Beauville and Yves Laszlo. Un lemme de descente. *Comptes Rendus de l'Academie des Sciences-Serie I-Mathematique*, 320(3):335–340, 1995.

[BMR06] Roman Bezrukavnikov, Ivan Mirković, and Dmitriy Rumynin. Singular localization and intertwining functors for reductive Lie algebras in prime characteristic. *Nagoya Mathematical Journal*, 183:1–55, 2006.

[BMRR08] Roman Bezrukavnikov, Ivan Mirković, Dmitriy Rumynin, and Simon Riche. Localization of modules for a semisimple Lie algebra in prime characteristic. *Annals of Mathematics*, pages 945–991, 2008.

[Bos14] Siegfried Bosch. *Lectures on formal and rigid geometry*, volume 2105. Springer, 2014.

[Bro93] Bram Broer. Line bundles on the cotangent bundle of the flag variety. *Inventiones mathematicae*, 113(1):1–20, 1993.

[Che11] Gaëtan Chenevier. On the infinite fern of Galois representations of unitary type. *Annales scientifiques de l'École Normale Supérieure*, Ser. 4, 44(6):963–1019, 2011.

[Col10] Pierre Colmez. Représentations de $\mathrm{GL}_2(\mathbb{Q}_p)$ et (φ, Γ)-modules. *Astérisque*, 330(281):509, 2010.

[Col16] Pierre Colmez. Représentations localement analytiques de $\mathrm{GL}_2(\mathbb{Q}_p)$ et (φ, Γ)-modules. *Represent. Theory*, 20:187–248, 2016.

[Col18] Pierre Colmez. Correspondance de Langlands locale p-adique et changement de poids. *Journal of the European Mathematical Society*, 21(3):797–838, 2018.

[Con06] Brian Conrad. Relative ampleness in rigid geometry. In *Annales de l'institut Fourier*, volume 56, pages 1049–1126, 2006.

[CT09] Brian Conrad and Michael Temkin. Non-archimedean analytification of algebraic spaces. *Journal of Algebraic Geometry*, 18(4):731–788, 2009.

[Din23] Yiwen Ding. Change of weights for locally analytic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. *arXiv:2307.04332*, 2023.

[Din24] Yiwen Ding. Towards the wall-crossing of locally $\mathbb{Q}_p$-analytic representations of $\mathrm{GL}_n(K)$ for a p-adic field K. *arXiv:2404.06315*, 2024.

[Dos12] Gabriel Dospinescu. Actions infinitésimales dans la correspondance de Langlands locale p-adique. *Mathematische Annalen*, 354(2):627–657, 2012.

[DPS20] Gabriel Dospinescu, Vytautas Paškūnas, and Benjamin Schraen. Infinitesimal characters in arithmetic families. *arXiv:2012.01041*, 2020.

[EGH23] Matthew Emerton, Toby Gee, and Eugen Hellmann. An introduction to the categorical p-adic Langlands program, 2023.

[Emm96] Ioannis Emmanouil. Mittag-Leffler condition and the vanishing of $\varprojlim^1$. *Topology*, 35(1):267–271, 1996.

[Fon04] Jean-Marc Fontaine. Arithmétique des représentations galoisiennes p-adiques. *Astérisque*, 295:1–115, 2004.

[Fru23] Martina Fruttidoro. Non-regular trianguline representations. *arXiv:2310.19499*, 2023.

[Gro60] Alexander Grothendieck. Éléments de géométrie algébrique : I. Le langage des schémas. *Publications Mathématiques de l'IHÉS*, 4:5–228, 1960.

[GV06] A Grothendieck and JL Verdier. *Théorie des Topos et Cohomologie Etale des Schémas. Séminaire de Géométrie Algébrique du Bois-Marie 1963-1964 (SGA 4): Tome 2*, volume 270. Springer, 2006.

[Har13] Robin Hartshorne. *Algebraic geometry*, volume 52. Springer Science & Business Media, 2013.

[Hel23] Eugen Hellmann. On the derived category of the Iwahori-Hecke algebra. *Compositio Mathematica*, 159(5):1042–1110, 2023.

[JLS21] Akash Jena, Aranya Lahiri, and Matthias Strauch. Translation functors for locally analytic representations. *arXiv preprint arXiv:2107.08493*, 2021.

[Ked09] Kiran S Kedlaya. Some new directions in p-adic hodge theory. *Journal de theorie des nombres de Bordeaux*, 21(2):285–300, 2009.

[Kie67] Reinhardt Kiehl. Theorem A und Theorem B in der nichtarchimedischen Funktionentheorie. *Inventiones mathematicae*, 2:256–273, 1967.

[Köp74] Ursula Köpf. Über eigentliche familien algebraischer varietäten über affinoiden räumen. *Schriftenreihe des Mathematischen Institutes Münster, 2.Serie. Heft 7*, 1974.

[KPX14] Kiran Kedlaya, Jonathan Pottharst, and Liang Xiao. Cohomology of arithmetic families of (φ, Γ)-modules. *Journal of the American Mathematical Society*, 27(4):1043–1115, 2014.

[Liu15] Ruochuan Liu. Triangulation of refined families. *Commentarii Mathematici Helvetici*, 90(4):831–904, 2015.

[Nak09] Kentaro Nakamura. Classification of two-dimensional split trianguline representations of p-adic fields. *Compositio Mathematica*, 145(4):865–914, 2009.

[Nak14] Kentaro Nakamura. Deformations of trianguline B-pairs and Zariski density of two dimensional crystalline representations. *Journal of mathematical sciences, the University of Tokyo*, 20(4):461–568, 2014.

[Poi10] Jérôme Poineau. Raccord sur les espaces de Berkovich. *Algebra & Number Theory*, 4(3):297–334, 2010.

[ST03] Peter Schneider and Jeremy Teitelbaum. Algebras of p-adic distributions and admissible representations. *Inventiones mathematicae*, 153:145–196, 2003.

[Sta24] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu`, 2024.

[Ste61] Robert Steinberg. A general Clebsch-Gordan theorem. *Bull. Amer. Math. Soc.*, 67(6):406–407, 1961.

[Str14] Oliver Straser. *On the geometric tensor functor for real reductive groups*. PhD thesis, Albert-Ludwigs-Universität Freiburg, 2014.

[Wu21] Zhixiang Wu. Local models for the trianguline variety and partially classical families. *arXiv:2103.03823*, 2021.

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