A NOTE ON PRESENTATIONS OF SUPERSINGULAR REPRESENTATIONS OF $\mathrm{GL}_2(F)$

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ABSTRACT. We prove that any smooth irreducible supersingular representation with central character of $\mathrm{GL}_2(F)$ is never of finite presentation when F is a finite field extension of $\mathbb{Q}_p$ such that $F \neq \mathbb{Q}_p$, extending a result of Schraen in [16] for quadratic extensions.

1. INTRODUCTION

Let p be a prime number. Let F be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$. Let $n \geq 2$ be an integer. Recent years, several progresses have been made on the study of representations of p-adic Lie groups on vector spaces over fields of characteristic p, motivated by the p-adic and mod-p Langlands programs. The classifications of mod-p irreducible admissible smooth representations of $\mathrm{GL}_n(F)$ in terms of supersingular representations was proved by Barthel-Livné for GL_2 ([3]) and by Herzig for general GL_n ([10]), which are now known for general reductive groups ([2]). Supersingular representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ was classified by Breuil and some mod-p Langlands correspondences appeared ([4]). Up to now, except $\mathrm{GL}_2(\mathbb{Q}_p)$ and some related groups such as $\mathrm{SL}_2(\mathbb{Q}_p)$ ([1],[6],[12]), supersingular representations for general groups (e.g. $\mathrm{GL}_3(\mathbb{Q}_p)$ or $\mathrm{GL}_2(F)$ when $F \neq \mathbb{Q}_p$) remain mysterious. Some complexity of classifications of supersingular representations of $\mathrm{GL}_2(F)$ when $F \neq \mathbb{Q}_p$ was shown by Breuil-Paškūnas's construction of supersingular representations ([5]). Daniel Le also constructed some non-admissible irreducible smooth mod-p representations for certain $\mathrm{GL}_2(F)$ ([13]).

Let $G = \mathrm{GL}_2(F), K = \mathrm{GL}_2(\mathcal{O})$ and Z be the center of G. Let π be an irreducible smooth representation of G over an algebraically closed characteristic p field k with central character. Then π contains a smooth irreducible sub-representation σ of subgroup KZ and there is a surjective morphism of G-representations $\mathrm{ind}_{KZ}^G\sigma \twoheadrightarrow \pi$ by the Frobenius reciprocity where $\mathrm{ind}_{KZ}^G\sigma$ denotes the compact induced representation. The representation π is called of finite presentation if the kernel of the surjection $\mathrm{ind}_{KZ}^G\sigma \twoheadrightarrow \pi$ is finitely generated as a $k[G]$-module. Such kind of finite presentation of representations of G when $G = \mathrm{GL}_2(\mathbb{Q}_p)$ are used by Colmez to construct a functor to get étale (φ, Γ)-modules from representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, which plays a key role in mod-p and p-adic Langlands correspondences for $\mathrm{GL}_2(\mathbb{Q}_p)$ ([7]). Vignéras constructed a generalized functor from representations of $\mathrm{GL}_2(F)$ of finite presentation to étale (φ, Γ)-modules of finite type ([17]). Unfortunately, Schraen proved in [16] that any smooth irreducible supersingular representation with central character of $\mathrm{GL}_2(F)$ is never of finite presentation when F is a quadratic field extension of $\mathbb{Q}_p$. The proof relies on a kind of coherent rings found by Emerton ([8]) and a criterion of finite presentation for representations of GL_2 by Hu (Theorem 1.3, [11]). In the note, we extend the result for any finite field extension F of $\mathbb{Q}_p$ such that $F \neq \mathbb{Q}_p$.

Theorem 1.1 (3.8)**.** *If $[F : \mathbb{Q}_p] \geq 2$, a smooth supersingular representation of $\mathrm{GL}_2(F)$ with central character is not of finite presentation.*

The proof firstly follows and simplifies the original arguments in [16]. Let $\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$ be the universal supersingular representation of G where T is the distinguished Hecke operator (cf. [3]). Let $L(\sigma)$ be the subspace of $\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$ generated by σ under the action of monoid $\begin{pmatrix} \varpi^{2\mathbb{N}} & \mathcal{O} \\ & 1 \end{pmatrix}$, where ϖ is a uniformizer of $\mathcal{O}$. Let $U := \begin{pmatrix} 1 & \mathcal{O} \\ & 1 \end{pmatrix}$ be the subgroup of unipotent upper-triangular matrices in $\mathrm{GL}_2(\mathcal{O})$. Using some arguments on modules over coherent rings (Lemma 3.1

and Lemma 3.2), we prove that π is not of finite presentation if the sub-module $L(\sigma)$ is not admissible, which means that the space $L(\sigma)^U$ of the U-invariants in $L(\sigma)$ is infinite-dimensional over k. The non-admissibility of $L(\sigma)$ is proved by explicitly finding invariant elements which is similar to works in [4], [15], [14] and [9]. A key observation is that the module structure of $L(\sigma)$ over the coherent ring guarantees that $\dim_k L(\sigma)^U = \infty$ if $\dim_k L(\sigma)^U \geq 2$. As a corollary, following [8] and [16], our result gives a uniform proof for the following fact.

Corollary 1.2 (4.5)**.** *For any smooth irreducible representation σ of KZ, the universal supersingular representation $ind_{KZ}^G\sigma/T(ind_{KZ}^G\sigma)$ of $GL_2(F)$ is not admissible if $F \neq \mathbb{Q}_p$.*

Organization of the note. In § 2, we recall basic facts on mod-p representations of $\mathrm{GL}_2(F)$ and Emerton's coherent rings. We prove the main result in § 3 with the proof for non-admissibility postponed to § 4.

Notations. We fix a uniformizer ϖ of F. Let k_F be the residue field of $\mathcal{O}$. Let $d = [F : \mathbb{Q}_p]$, $f = [k_F : \mathbb{F}_p]$, $e = d/f$ and $q = p^f$. Let $G = \mathrm{GL}_2(F)$, $K = \mathrm{GL}_2(\mathcal{O})$ and Z be the center of G. Let K_1 be the kernel of the reduction map $K \to \mathrm{GL}_2(k_F)$. Let $U = \left\{\begin{pmatrix} 1 & a \\ & 1\end{pmatrix}, a \in \mathcal{O}\right\}$ and $\alpha = \begin{pmatrix} \varpi & \\ & 1\end{pmatrix}$. Let k be an algebraically closed field of characteristic p. We identify $k_F = \mathbb{F}_q$ and fix an embedding $k_F \hookrightarrow k$. All the representations in the note are on vector spaces over k.

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2. Preliminary on representations and coherent rings

Mod-p representations of $\mathbf{GL}_2$. We recall some results and notations in [3] and [4]. Let π be a smooth irreducible representation of G with central character over k. Then π contains an irreducible sub-KZ-representation σ of KZ. Let $\mathrm{ind}_{KZ}^G\sigma$ be the compactly induced representation: the representation space consists of functions $f : G \to \sigma$ such that f is compactly supported modulo KZ and $f(k\cdot) = k.f(\cdot)$ for any $k \in KZ$ and the action of G is given by right translations. There is a distinguished element $T \in \mathrm{End}_G(\mathrm{ind}_{KZ}^G\sigma)$ which generates the Hecke algebra. By the definition and the classification in [3], π is supersingular if and only if there exists a surjection $\mathrm{ind}_{KZ}^G\sigma \twoheadrightarrow \pi$ induced by an inclusion $\sigma \hookrightarrow \pi|_{KZ}$ and the Frobenius reciprocity such that the surjection factors through a map

$$\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma) \twoheadrightarrow \pi$$

for some or every such σ.

If $0 \leq r \leq p-1$ is an integer, let Sym^r be the r-th symmetric power of the standard representation of $\mathrm{GL}_2(\mathbb{F}_q)$ on two-dimensional space k^2 via the embedding $\mathbb{F}_q \hookrightarrow k$. If $\vec{r} = (r_0, \cdots, r_{f-1}) \in \mathbb{Z}^f$ with $0 \leq r_j \leq p-1$ for any $0 \leq j \leq f-1$, we get a representation $\mathrm{Sym}^{\vec{r}} := \otimes_{j=0}^{f-1}\mathrm{Sym}^{r_j} \circ \mathrm{Fr}^j$, where Fr denotes the automorphism of $\mathrm{GL}_2(\mathbb{F}_q)$ induced by the Frobenius automorphism of $\mathbb{F}_q$. If $\vec{a}, \vec{b} \in \mathbb{Z}^f$, we say $\vec{a} \leq \vec{b}$ if $a_j \leq b_j$ for any $j = 0, \cdots, f-1$. The representation $\mathrm{Sym}^{\vec{r}}$ has a model consisting of homogeneous polynomials spanned by a basis $\{\otimes_{j=0}^{f-1} x_j^{r_j - i_j} y_j^{i_j}\}_{0 \leq \vec{i} \leq \vec{r}}$. The group action is given by

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix} \cdot \otimes_{j=0}^{f-1} x_j^{r_j-i_j} y_j^{i_j} = \otimes_{j=0}^{f-1}(a^{p^j}x_j + c^{p^j}y_j)^{r_j-i_j}(b^{p^j}x_j + d^{p^j}y_j)^{i_j},$$

for any $0 \leq \vec{i} \leq \vec{r}$, $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_q)$. We abbreviate $x^{\vec{r}-\vec{i}}y^{\vec{i}} := \otimes_{j=0}^{f-1} x_j^{r_j-i_j} y_j^{i_j}$. If $\chi : \mathbb{F}_q^\times \to k^\times$ is a character of $\mathbb{F}_q^\times$, $\chi \circ \det$ is a character of $\mathrm{GL}_2(\mathbb{F}_q)$. We can naturally inflate the representation

$(\chi \circ \det) \otimes \mathrm{Sym}^{\vec{r}}$ of $\mathrm{GL}_2(\mathbb{F}_q)$ to a representation of K by letting K_1 act trivially. Then the smooth irreducible KZ-representation σ is isomorphic to $(\chi \circ \det) \otimes \mathrm{Sym}^{\vec{r}}$ when restricted to K for a unique $\chi : \mathbb{F}_q^\times \to k^\times$ and $\vec{r}$ as above and the action of $\begin{pmatrix} \varpi & \\ & \varpi \end{pmatrix} \in Z$ on σ is given by a scalar $\nu \in k^\times$.

If $g \in G, w \in \sigma$, let $[g, w] \in \mathrm{ind}_{KZ}^G \sigma$ be the element given by

$$[g, w](g') = \begin{cases} g'g.w & \text{if } g'g \in KZ, \\ 0 & \text{if } g'g \notin KZ. \end{cases}$$

Then $g'.[g, w] = [g'g, w], \forall g', g \in G, w \in \sigma$. If $S \subset G$ is a subset, let $[S, \sigma]$ be the subspace of $\mathrm{ind}_{KZ}^G \sigma$ spanned by $[g, w], w \in \sigma, g \in S$.

If $\lambda \in \mathbb{F}_q$, we let $[\lambda]$ be the Teichmüller lift of λ in F. For any integer $n \geq 1$, the set $I_n := \{[\lambda_0] + \varpi[\lambda_1] + \cdots + \varpi^{n-1}[\lambda_{n-1}], \lambda_i \in \mathbb{F}_q\}$ is a complete set of representatives of $\mathcal{O}/\varpi^n\mathcal{O}$. We define $I_0 = \{0\}$. If $\lambda = [\lambda_0] + \varpi[\lambda_1] + \cdots + \varpi^{n-1}[\lambda_{n-1}] \in I_n$, let $[\lambda]_{n-1} := \lambda - \varpi^{n-1}[\lambda_{n-1}] \in I_{n-1}$. If $\vec{i} \in \mathbb{Z}^f, \lambda \in \mathbb{F}_q$, we use the notation $\lambda^{\vec{i}} := \lambda^{\sum_{0 \leq j \leq f-1} p^j i_j}$. The element $\left[\begin{pmatrix} \varpi^n & \lambda \\ & 1 \end{pmatrix}, \sum_{\vec{0} \leq \vec{i} \leq \vec{r}} u_{\vec{i}} x^{\vec{r}-\vec{i}} y^{\vec{i}}\right] \in \mathrm{ind}_{KZ}^G \sigma$ where $u_{\vec{i}} \in k$ for any $\vec{i}, n \in \mathbb{N}, \lambda \in I_n$. The action of the operator T on the element is calculated as in [4] (or see Proposition 2.1, [9]). If $n \geq 1, \mu \in I_n$,

$$\begin{aligned}
T\left(\left[\begin{pmatrix} \varpi^n & \mu \\ & 1 \end{pmatrix}, \sum_{0 \leq \vec{i} \leq \vec{r}} u_{\vec{i}} x^{\vec{r}-\vec{i}} y^{\vec{i}}\right]\right) &= \sum_{\lambda \in \mathbb{F}_q} \left[\begin{pmatrix} \varpi^{n+1} & \mu + \varpi^n[\lambda] \\ & 1 \end{pmatrix}, (\sum_{0 \leq \vec{i} \leq \vec{r}} u_{\vec{i}} (-\lambda)^{\vec{i}}) x^{\vec{r}}\right] \\
&\quad + \nu \left[\begin{pmatrix} \varpi^{n-1} & [\mu]_{n-1} \\ & 1 \end{pmatrix}, u_{\vec{r}} \otimes_{j=0}^{f-1} (\mu_{n-1}^{p^j} x_j + y_j)^{r_j}\right], \\
T\left(\left[\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \sum_{0 \leq \vec{i} \leq \vec{r}} u_{\vec{i}} x^{\vec{r}-\vec{i}} y^{\vec{i}}\right]\right) &= \sum_{\lambda \in \mathbb{F}_q} \left[\begin{pmatrix} \varpi & [\lambda] \\ & 1 \end{pmatrix}, (\sum_{0 \leq \vec{i} \leq \vec{r}} u_{\vec{i}} (-\lambda)^{\vec{i}}) x^{\vec{r}}\right] \\
&\quad + \left[\begin{pmatrix} 1 & \\ & \varpi \end{pmatrix}, u_{\vec{r}} y^{\vec{r}}\right].
\end{aligned} \tag{2.1}$$

A class of coherent rings. We now recall some results in [8] and [16] on a type of coherent rings and their applications on representations of GL_2. Assume A is a complete regular local ring of dimension d with residue field k and maximal ideal $\mathfrak{m}$. Assume $\phi : A \to A$ is a local flat ring endomorphism of A and assume ϕ is equal to the identity map on k after reduction modulo $\mathfrak{m}$. We let $A[X]_\phi$ be the ring of polynomials in variable X with commutative relation $Xa = \phi(a)X, \forall a \in A$. By Proposition 1.3 in [8], $A[X]_\phi$ is a coherent ring which means that any finitely generated submodule of a finitely presented left $A[X]_\phi$-module is finitely presented.

Modulo $\mathfrak{m}$, we get a ring morphism $A[X]_\phi \to k[X]$. If M is a left $A[X]_\phi$-module, there are natural isomorphisms $\mathrm{Tor}_i^{A[X]_\phi}(k[X], M) \simeq \mathrm{Tor}_i^A(k, M)$ for all $i \geq 0$ (Lemma 2.1, [8]). The isomorphisms equip the k-spaces $\mathrm{Tor}_i^A(k, M)$ $k[X]$-module structures. If M is a finitely presented $A[X]_\phi$-module, then for any $i \geq 0$, $\mathrm{Tor}_i^A(k, M)$ is a finitely generated $k[X]$-module (Proposition 2.2, [8]).

An A-module is called smooth if any finitely generated submodule is Artinian. An $A[X]_\phi$-module is called smooth if the underlying A-module is smooth. If M is an A-module, we let $M[\mathfrak{m}] = \{x \in M \mid mx = 0, \forall m \in \mathfrak{m}\}$. There is a non-canonical isomorphism between functor $M \mapsto M[\mathfrak{m}]$ and functor $M \mapsto \mathrm{Tor}_d^A(k, M)$. An A-module M is called admissible if it is smooth and $M[\mathfrak{m}] \simeq \mathrm{Tor}_d^A(k, M)$ is finite-dimensional over k. An $A[X]_\phi$-module is called admissible if the underlying A-module is admissible.

From now on, we let $A := k[[U]] = \varprojlim k[U/N]$, where N ranges over all open normal subgroups of U, be the Iwasawa algebra of U. Then $A \simeq k[[X_1, \cdots, X_d]]$ the ring of formal power series in d variables with maximal ideal $\mathfrak{m} = (X_1, \cdots, X_d)$. The action of $\alpha = \begin{pmatrix} \varpi & \\ & 1 \end{pmatrix}$ on $U : u \mapsto \alpha u \alpha^{-1}$ induces a flat local morphism $\phi : A \to A$. If Π is a smooth representation of U, Π is naturally a smooth A-module and $\Pi^U = \Pi[\mathfrak{m}]$. Thus the representation Π is an admissible A-module if and only if Π is an admissible U-representation. Any representation Π of monoid $\begin{pmatrix} \varpi^{\mathbb{N}} & \mathcal{O} \\ & 1 \end{pmatrix}$ is now naturally an $A[X]_\phi$-module where X acts by the action of α on Π.

Let σ be an irreducible smooth representation of KZ. For any $n \geq 0$, let $R_n(\sigma) := \left[\begin{pmatrix} \varpi^n & \mathcal{O} \\ & 1 \end{pmatrix}, \sigma \right]$ which is a sub-A-module of $\mathrm{ind}_{KZ}^G \sigma$. For any $k \in \mathbb{N}$, we let

$$I_{\geq k}(\sigma) := \bigoplus_{n \geq k} R_n(\sigma), \quad I^e_{\geq k}(\sigma) := \bigoplus_{n \geq k, 2|n} R_n(\sigma), \quad I^o_{\geq k}(\sigma) := \bigoplus_{n \geq k, 2\nmid n} R_n(\sigma),$$

be subspaces of $\mathrm{ind}_{KZ}^G \sigma$. We let $\phi_2 := \phi^2 : A \to A$. We have (Lemma 2.11, [16])

$$I^e_{\geq 0}(\sigma) \simeq A[X]_{\phi_2} \otimes_A \sigma, \quad I^o_{\geq 1}(\sigma) \simeq A[X]_{\phi_2} \otimes_A R_1(\sigma)$$

as $A[X]_{\phi_2}$-modules.

By the formula of the operator T (2.1), we have $T(R_n(\sigma)) \subset R_{n+1}(\sigma) \oplus R_{n-1}(\sigma)$ if $n \geq 1$. Hence $T(I_{\geq 1}(\sigma)) \subset I_{\geq 0}(\sigma)$ and $T(I^o_{\geq 1}(\sigma)) \subset I^e_{\geq 0}(\sigma)$, etc. We decompose $T|_{I_{\geq 1}(\sigma)} = T_+ + T_-$ by the decomposition $T|_{R_n(\sigma)} = T_+|_{R_n(\sigma)} + T_-|_{R_n(\sigma)}$, where $T_+|_{R_n(\sigma)} : R_n(\sigma) \to R_{n+1}(\sigma)$ and $T_-|_{R_n(\sigma)} : R_n(\sigma) \to R_{n-1}(\sigma)$ are compositions of the projections to the direct sum factors of $R_{n+1}(\sigma) \oplus R_{n-1}(\sigma)$ and $T|_{R_n(\sigma)}$, for all $n \geq 1$.

Let $L(\sigma) := I^e_{\geq 0}(\sigma)/T(I^o_{\geq 1}(\sigma))$. Then $L(\sigma)$ is an $A[X]_{\phi_2}$-module. The following proposition is essentially Proposition 2.23 in [16] which we recall the proof.

Proposition 2.1. *$Tor_0^A(k, L(\sigma)) = 0$. The $k[X]$-torsion part of $Tor_d^A(k, L(\sigma))$ is isomorphic to $k = k[X]/(X)$ and coincides with the image of $Tor_d^A(k, \sigma)$ via the morphism $\sigma \hookrightarrow I^e_{\geq 0}(\sigma) \twoheadrightarrow L(\sigma)$.*

Proof We have an exact sequence

$$0 \to \mathrm{Tor}_d^A(k, I^o_{\geq 1}(\sigma)) \overset{\mathrm{Tor}_d^A(T)}{\to} \mathrm{Tor}_d^A(k, I^e_{\geq 0}(\sigma)) \to \mathrm{Tor}_d^A(k, L(\sigma)) \to \mathrm{Tor}_{d-1}^A(k, I^o_{\geq 1}(\sigma)) \cdots$$
$$\cdots \to \mathrm{Tor}_0^A(k, I^o_{\geq 1}(\sigma)) \overset{\mathrm{Tor}_0^A(T)}{\to} \mathrm{Tor}_0^A(k, I^e_{\geq 0}(\sigma)) \to \mathrm{Tor}_0^A(k, L(\sigma)) \to 0.$$

And $\mathrm{Tor}_i^A(k, I^e_{\geq 0}(\sigma)) \simeq \oplus_{k \geq 0} \mathrm{Tor}_i^A(k, R_{2k}(\sigma))$, $\mathrm{Tor}_i^A(k, I^o_{\geq 1}(\sigma)) \simeq \oplus_{k \geq 0} \mathrm{Tor}_i^A(k, R_{2k+1}(\sigma))$ for $i \in \mathbb{N}$.

By Lemma 2.12 in [16], $\mathrm{Tor}_0^A(T_+) = 0$, $\mathrm{Tor}_0^A(T_-) = \mathrm{Tor}_0^A(T)$ and $\mathrm{Tor}_0^A(T_-)$ sends each $\mathrm{Tor}_0^A(k, R_{2k+1}(\sigma))$ onto $\mathrm{Tor}_0^A(k, R_{2k}(\sigma))$. Hence $\mathrm{Tor}_0^A(T)$ in the above diagram is a surjection and $\mathrm{Tor}_0^A(k, L(\sigma)) = 0$. Since $\mathrm{Tor}_{d-1}^A(k, I^o_{\geq 1}(\sigma)) \simeq \mathrm{Tor}_{d-1}^A(k, A[X]_{\phi_2} \otimes_A (A \otimes_{\phi, A} \sigma)) \simeq k[X] \otimes_k \mathrm{Tor}_{d-1}^A(k, A \otimes_{\phi, A} \sigma)$ by Proposition 1.4 in [16], the $k[X]$-module $\mathrm{Tor}_{d-1}^A(k, I^o_{\geq 1}(\sigma))$ is torsion free. Hence $\mathrm{Tor}_d^A(k, L(\sigma))_{tors} = \mathrm{coker}(\mathrm{Tor}_d^A(T))_{tors}$. By Lemma 2.12 in [16] again, $\mathrm{Tor}_d^A(T_-) = 0$ and $\mathrm{Tor}_d^A(T_+)$ sending $\mathrm{Tor}_d^A(k, R_{2k+1}(\sigma))$ to $\mathrm{Tor}_d^A(k, R_{2k+2}(\sigma))$ is an isomorphism. Thus the image of $\mathrm{Tor}_d^A(T)$ in $\mathrm{Tor}_d^A(k, I^e_{\geq 0}(\sigma))$ is $\oplus_{k \geq 1} \mathrm{Tor}_d^A(k, R_{2k}(\sigma))$. Since $R_0(\sigma) = \sigma$, $\mathrm{Tor}_d^A(k, L(\sigma))_{tors}$ coincides with the image of $\mathrm{Tor}_d^A(k, \sigma)$ via the map $\sigma \hookrightarrow I^e_{\geq 0}(\sigma) \twoheadrightarrow L(\sigma)$ in $L(\sigma)$. Finally, $\mathrm{Tor}_d^A(k, \sigma) \simeq \sigma^U$ is one-dimensional over k by Lemma 2 in [3]. $\square$

We recall the following key lemma on smooth finitely presented $A[X]_\phi$-modules in [16].

Lemma 2.2 ([16], Lemma 1.13). *Let M be a smooth finitely presented $A[X]_\phi$-module. Then there exists an increasing sequence of sub-$A[X]_\phi$-modules $(M_i)_{i\geq 0}$, a sequence of finite-dimensional k-vector spaces $(V_i)_{i\geq 0}$ such that there exist isomorphisms $M_{i+1}/M_i \simeq A[X]_\phi \otimes_A V_i$ as $A[X]_\phi$-modules, and if we let $\widetilde{M} = \cup_i M_i$, then $Tor_d^A(k, M)_{tors} \simeq Tor_d^A(k, M/\widetilde{M})$. In particular, $M/\widetilde{M}$ is admissible and each M_i is of finite presentation.*

3. Presentations of supersingular representations

We prove some lemmas on $A[X]_\phi$-modules.

Lemma 3.1. *Let M be a non-zero, smooth, finitely presented $A[X]_\phi$-module. Assume that $Tor_d^A(k, M)$ is a torsion free $k[X]$-module. Then $Tor_0^A(k, M)$ is infinite-dimensional over k.*

Proof By Lemma 2.2, we can find an increasing sequence of sub-$A[X]_\phi$-modules $(M_i)_{i\geq 0}$, a sequence of finite-dimensional k-vector spaces $(V_i)_{i\geq 0}$ such that there exist isomorphisms $M_{i+1}/M_i \simeq A[X]_\phi \otimes_A V_i$ of $A[X]_\phi$-modules with $M_0 = 0$, and if we let $\widetilde{M} = \cup_i M_i$, then $\mathrm{Tor}_d^A(k, M)_{tors} \simeq \mathrm{Tor}_d^A(k, M/\widetilde{M})$. Thus $\mathrm{Tor}_d^A(k, M/\widetilde{M}) = 0$ by assumptions. Hence $M = \widetilde{M}$ by Lemma 1.8 in [16]. Since M is finitely generated, there exists a minimal $n \in \mathbb{N}$ such that $M = M_n$. Since M is non-zero, we have $n \geq 1$ and $M_n \neq M_{n-1}$. We have a surjection

$$M \twoheadrightarrow M/M_{n-1} \simeq A[X]_\phi \otimes_A V_{n-1}.$$

Thus we have a surjection

$$\mathrm{Tor}_0^A(k, M) \twoheadrightarrow \mathrm{Tor}_0^A(k, A[X]_\phi \otimes_A V_{n-1}).$$

But by Proposition 1.4 and Example 1.6 in [16], $\mathrm{Tor}_0^A(k, A[X]_\phi \otimes_A V_{n-1}) \simeq k[X] \otimes \mathrm{Tor}_0^A(k, V_{n-1})$ is a free $k[X]$-module of rank $\dim_k V_{n-1}$. Assume that $\mathrm{Tor}_0^A(k, M)$ is finite-dimensional over k. Then V_{n-1} is zero by the surjection above. This contradicts that $M_n \neq M_{n-1}$. Hence $\mathrm{Tor}_0^A(k, M)$ is infinite-dimensional over k. □

Lemma 3.2. *Let M be a smooth, finitely presented $A[X]_\phi$-module and N be a non-zero sub-$A[X]_\phi$-module of M. Assume that M/N is finitely presented and admissible, and $Tor_d^A(k, N)$ is torsion free. Then $Tor_0^A(k, M)$ is infinite-dimensional over k.*

Proof Since M/N and M are finitely presented, by the coherence of $A[X]_\phi$ (Proposition 1.3, [8]), N is of finite presentation. Thus by Lemma 3.1, $\mathrm{Tor}_0^A(k, N)$ is infinite-dimensional over k. Consider the long exact sequence

$$\cdots \to \mathrm{Tor}_1^A(k, M/N) \to \mathrm{Tor}_0^A(k, N) \to \mathrm{Tor}_0^A(k, M) \to \mathrm{Tor}_0^A(k, M/N) \to 0.$$

Since M/N is admissible, by Corollary 1.12 in [16], $\mathrm{Tor}_1^A(k, M/N)$ and $\mathrm{Tor}_0^A(k, M/N)$ are finite-dimensional over k. Since $\mathrm{Tor}_0^A(k, N)$ is infinite-dimensional over k, so is $\mathrm{Tor}_0^A(k, M)$. □

Definition 3.3. *A smooth representation π of G is called of finite presentation if there exists an irreducible smooth representation σ of KZ and a surjection*

$$ind_{KZ}^G \sigma \twoheadrightarrow \pi$$

such that the kernel is finitely generated as a $k[G]$-module.

Remark 3.4. *By Proposition 4.4 in* [11], *if π is of finite presentation, then for all smooth finite-dimensional sub-KZ-representation σ of π which generates the G-representation π, the kernel of the surjection $ind_{KZ}^G \sigma \twoheadrightarrow \pi$ is finitely generated as a $k[G]$-module.*

Remark 3.5. *If $F = \mathbb{Q}_p$, then by the classifications in* [3] *and* [4], *any irreducible representation of $GL_2(\mathbb{Q}_p)$ with central character is of finite presentation.*

Assume π is a smooth irreducible representation of G with central character, and $\sigma \subset \pi$ is an irreducible smooth sub-KZ-representation. Let $I^+(\pi,\sigma) := \begin{pmatrix} \varpi^{\mathbb{N}} & \mathcal{O} \\ & 1 \end{pmatrix} \sigma \subset \pi$ be the $A[X]_\phi$-submodule of π generated by σ. Then $I^+(\pi,\sigma)$ is the image of $I_{\geq 0}(\sigma)$ in π via the map $\mathrm{ind}_{KZ}^G \sigma \twoheadrightarrow \pi$. We recall the following result of Yongquan Hu.

Theorem 3.6 ([11], Theorem 1.3)**.** *If π is of finite presentation, then $I^+(\pi,\sigma)^U$ is a finite-dimensional k-vector space.*

We will prove the following theorem in § 4.

Theorem 3.7. *The A-module $L(\sigma)$ is not admissible if $[F:\mathbb{Q}_p] \geq 2$. In particular, the $k[X]$-module $Tor_d^A(k, L(\sigma))$ is not torsion.*

Now assuming Theorem 3.7, we prove the main theorem.

Theorem 3.8. *If π is a smooth supersingular representation of $GL_2(F)$ with central character, then π is not of finite presentation when $[F:\mathbb{Q}_p] \geq 2$.*

Proof We can find a surjection $\mathrm{ind}_{KZ}^G(\sigma)/(T) \twoheadrightarrow \pi$ for some irreducible smooth sub-KZ-representation σ of π by the definition of supersingular representations. Let $I^+(\pi,\sigma)$ be the $A[X]_\phi$-submodule of π generated by σ and let $M(\pi,\sigma)$ be the $A[X]_{\phi_2}$-submodule of π generated by σ. Then $M(\pi,\sigma) \subset I^+(\pi,\sigma)$. The map of $A[X]_{\phi_2}$-modules $I^e_{\geq 0}(\sigma) \hookrightarrow \mathrm{ind}_{KZ}^G \sigma \to \pi$ factors through $L(\sigma) \to \pi$ with image $M(\pi,\sigma)$. Let $N(\pi,\sigma)$ be the kernel of the morphism $L(\sigma) \to M(\pi,\sigma)$ of $A[X]_{\phi_2}$-modules. We have an exact sequence

$$0 \to \mathrm{Tor}_d^A(k, N(\pi,\sigma)) \to \mathrm{Tor}_d^A(k, L(\sigma)) \to \mathrm{Tor}_d^A(k, M(\pi,\sigma)).$$

By Proposition 2.1, $\mathrm{Tor}_d^A(k, L(\sigma))_{tors}$ is generated by the image of $\sigma^U \simeq \mathrm{Tor}_d^A(k,\sigma)$ via the map $\sigma \to I^e_{\geq 0}(\sigma) \twoheadrightarrow L(\sigma)$. The non-zero composition map $\sigma \to L(\sigma) \to M(\pi,\sigma)$ induces morphisms $\mathrm{Tor}_d^A(k,\sigma) \xrightarrow{\sim} \mathrm{Tor}_d^A(k, L(\sigma))_{tors} \to \mathrm{Tor}_d^A(k, M(\pi,\sigma))$. The composition $\sigma \to M(\pi,\sigma)$ is injective since σ is irreducible. Since $\mathrm{Tor}_d^A(k,-)$ is left exact, we get an injection $\mathrm{Tor}_d^A(k, L(\sigma))_{tors} \hookrightarrow \mathrm{Tor}_d^A(k, M(\pi,\sigma))$. Then $\mathrm{Tor}_d^A(k, N(\pi,\sigma))$ must be a torsion free $k[X]$-module.

Now if π is finitely presented, $M(\pi,\sigma) \subset I^+(\pi,\sigma)$ is admissible by Hu's result (Theorem 3.6). Since $M(\pi,\sigma)$ is generated by σ, it is a finitely generated $A[X]_{\phi_2}$-module. Moreover, the proof of Theorem 2.24 in [16] shows that $M(\pi,\sigma)$ is of finite presentation ($M(\pi,\sigma)$ is stable under the action of $H = \begin{pmatrix} \mathcal{O}^\times & \\ & 1 \end{pmatrix}$, then use Lemma 2.6 in [16]). If $N(\pi,\sigma) \neq 0$, then all the assumptions in Lemma 3.2 are satisfied if we take $M = L(\sigma)$ and $N = N(\pi,\sigma)$. Remark that here $L(\sigma), N(\pi,\sigma)$ are modules over the coherent ring $A[X]_{\phi_2}$ rather than $A[X]_\phi$, but Lemma 3.2 holds true for $A[X]_{\phi_2}$. Thus by Lemma 3.2, $\mathrm{Tor}_0^A(k, L(\sigma))$ has infinite dimension over k, which contradicts that $\mathrm{Tor}_0^A(k, L(\sigma)) = 0$ (Proposition 2.1)! Hence $N(\pi,\sigma) = 0$. Then $L(\sigma) \simeq M(\pi,\sigma)$ is admissible. This contradicts Theorem 3.7! Hence π is not of finite presentation. □

4. Non-admissibility

Assume $\sigma = \mathrm{Sym}^{\vec{r}} \otimes (\chi \circ \det)$, where $\vec{r} = (r_0, \cdots, r_{f-1})$ such that $0 \leq r_0, \cdots, r_{f-1} \leq p-1$, is an irreducible representation of KZ with $\varpi \in Z$ acting on σ as a scalar $\nu \in k^\times$. Recall that

$$R_1(\sigma) = \bigoplus_{\mu \in \mathbb{F}_q} \left[\begin{pmatrix} \varpi & [\mu] \\ & 1 \end{pmatrix}, \sigma \right], \quad R_2(\sigma) = \bigoplus_{\mu,\lambda \in \mathbb{F}_q} \left[\begin{pmatrix} \varpi^2 & [\mu] + \varpi[\lambda] \\ & 1 \end{pmatrix}, \sigma \right].$$

For any $\mu \in \mathbb{F}_q, u_{\vec{i}} \in k, \vec{i} \in \mathbb{Z}^f, \vec{0} \leq \vec{i} \leq \vec{r}$, the operators $T_\pm$ act on $\left[\begin{pmatrix} \varpi & [\mu] \\ & 1 \end{pmatrix}, \sum_{0 \leq \vec{i} \leq \vec{r}} u_{\vec{i}} x^{\vec{r}-\vec{i}} y^{\vec{i}}\right] \in R_1(\sigma)$ by the formulas (see (2.1)):

$$T_+\left(\left[\begin{pmatrix} \varpi & [\mu] \\ & 1 \end{pmatrix}, \sum_{0 \leq \vec{i} \leq \vec{r}} u_{\vec{i}} x^{\vec{r}-\vec{i}} y^{\vec{i}}\right]\right) = \sum_{\lambda \in \mathbb{F}_q}\left[\begin{pmatrix} \varpi^2 & [\mu]+\varpi[\lambda] \\ & 1 \end{pmatrix}, (\sum_{0 \leq \vec{i} \leq \vec{r}} u_{\vec{i}}(-\lambda)^{\vec{i}}) x^{\vec{r}}\right] \tag{4.1}$$

$$T_-\left(\left[\begin{pmatrix} \varpi & [\mu] \\ & 1 \end{pmatrix}, \sum_{0 \leq \vec{i} \leq \vec{r}} u_{\vec{i}} x^{\vec{r}-\vec{i}} y^{\vec{i}}\right]\right) = \nu\left[\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, u_{\vec{r}} \otimes_{j=0}^{f-1} (\mu^{p^j} x_j + y_j)^{r_j}\right]. \tag{4.2}$$

Proof of Theorem 3.7 We need to prove that $L(\sigma)^U$ is infinite-dimensional over k. By Proposition 2.1, the torsion part of the $k[X]$-module $\mathrm{Tor}_d^A(k, L(\sigma)) \simeq L(\sigma)^U$ has only dimension 1. If $\dim_k L(\sigma)^U \geq 2$, the free part of the $k[X]$-module $\mathrm{Tor}_d^A(k, L(\sigma))$ can not be zero and then $\mathrm{Tor}_d^A(k, L(\sigma))$ is infinite-dimensional over k since a non-zero free $k[X]$-module is infinite-dimensional over k. So we only need to prove that $\dim_k L(\sigma)^U \geq 2$ to show that $L(\sigma)$ is not an admissible A-module. We will prove

Lemma 4.1. *If $[F : \mathbb{Q}_p] \geq 2$, there exists an element $g \in R_2(\sigma)$ such that $g \notin T_+R_1(\sigma)$ and $ug - g \in TR_1'(\sigma)$ for any $u \in U$, where $R_1'(\sigma)$ is the kernel of $T_-|_{R_1(\sigma)}$.*

Now assume there exists an element g as in Lemma 4.1. Then the image of g in $L(\sigma)$ lies in $L(\sigma)^U$ since $ug - g \in TR_1'(\sigma) \subset TI_{\geq 1}^o(\sigma)$ which is zero in $L(\sigma) = I_{\geq 0}^e(\sigma)/TI_{\geq 1}^o(\sigma)$ for any $u \in U$. We claim that the image of g doesn't lie in the image of $R_0(\sigma)$ in $L(\sigma)$. Otherwise there exist $a \in R_0(\sigma), x \in I_{\geq 1}^o(\sigma)$ such that $g - a = Tx$. Assume $x = \sum_{k \in \mathbb{N}} x_{2k+1}$, where each $x_{2k+1} \in R_{2k+1}(\sigma)$ and there are only finitely many k such that $x_{2k+1} \neq 0$. Since $g \notin T_+R_1(\sigma)$, $g \neq 0$ and we may assume $x \neq 0$. Let k_0 be the maximal integer such that $x_{2k_0+1} \neq 0$. Then $Tx = T_-(x_1) + \sum_{k=0}^{k_0}(T_+(x_{2k+1}) + T_-(x_{2k+3})) \in R_0(\sigma) \oplus (\oplus_{k=0}^{k_0} R_{2k+2}(\sigma))$. Since $Tx = g - a \in R_0(\sigma) \oplus R_2(\sigma)$, if $k_0 \neq 0$, $T_+(x_{2k_0+1}) = 0 \in R_{2k_0+2}(\sigma)$. This contradicts that T_+ is injective (Lemma 2.12 in [16]) and x_{2k_0+1} is not 0. If $k_0 = 0$, then $g = T_+(x_1) \in T_+R_1(\sigma)$, which contradicts our choice of g in the Lemma 4.1. Hence the image of g in $L(\sigma)$ doesn't lie in the image of $R_0(\sigma)$ in $L(\sigma)$. Thus the image of σ^U and g in $L(\sigma)$ span a two-dimensional subspace of $L(\sigma)^U$. This proves that $\dim_k(L(\sigma)) \geq 2$ and $L(\sigma)$ is not admissible. $\square$

Before the proof of Lemma 4.1, we remark the following simple facts.

Lemma 4.2. *Let $F = \sum_i a_i X^i \in k[X]$ be a polynomial of degree no more than $q-1$, then $\sum_{t \in \mathbb{F}_q} F(t) = -a_{q-1}$.*

Lemma 4.3 ([15], Lemma 2.2)**.** *For any $a, b \in \mathbb{F}_q$, $[a] + [b] \equiv [a+b] + \varpi^e[P(a,b)] \mod \varpi^{e+1}$, where $P(a,b) = \frac{a^{q^e} + b^{q^e} - (a+b)^{q^e}}{\varpi^e}$.*

Proof of Lemma 4.1 Our method is to find a concrete required element g in all possible cases. We remark firstly that by (4.1), $T_+R_1(\sigma)$ is spanned (over k) by elements

$$\sum_{\lambda \in \mathbb{F}_q}\left[\begin{pmatrix} \varpi^2 & [\mu]+\varpi[\lambda] \\ & 1 \end{pmatrix}, \lambda^{\vec{i}} x^{\vec{r}}\right]$$

where $\vec{0} \leq \vec{i} \leq \vec{r}$ and $\mu \in \mathbb{F}_q$. Moreover $\sum_{\lambda \in \mathbb{F}_q}\left[\begin{pmatrix} \varpi^2 & [\mu]+\varpi[\lambda] \\ & 1 \end{pmatrix}, \lambda^{\vec{i}} x^{\vec{r}}\right]$ lies in $T_+R_1'(\sigma)$ if $\vec{i} < \vec{r}$ by (4.1) and (4.2), here $\vec{i} < \vec{r}$ means $\vec{i} \leq \vec{r}$ and $\vec{i} \neq \vec{r}$. Since $T_\pm$ are U-equivariant, $R_1'(\sigma)$ and $T_+R_1'(\sigma)$ are stable under the action of U. Moreover, $\alpha^3 U \alpha^{-3} = \begin{pmatrix} 1 & \varpi^3\mathcal{O} \\ & 1 \end{pmatrix}$ acts trivially on $R_2(\sigma)$.

1) Assume F is ramified over $\mathbb{Q}_p$ with $e \geq 2$. We have

$$[a] + [b] \equiv [a+b] \mod \varpi^2,$$

by Lemma 4.3.

If $\dim_k(\sigma) > 1$, there exists j_0 such that $r_{j_0} \geq 1$. Let $\vec{i'} = (i'_0, \cdots, i'_{f-1}) \in \mathbb{Z}^f$ where $i'_j = 0$ if $j \neq j_0$ and $i'_{j_0} = 1$. Then $\vec{i'} \leq \vec{r}$. We take

$$g = \sum_{\mu,\lambda \in \mathbb{F}_q} \left[\begin{pmatrix} \varpi^2 & [\mu] + \varpi[\lambda] \\ & 1 \end{pmatrix}, x^{\vec{r}-\vec{i'}} y^{\vec{i'}} \right].$$

Then $g \notin T_+ R_1(\sigma)$. For $a \in \mathbb{F}_q$, we calculate that

$$\begin{aligned}
\begin{pmatrix} 1 & \varpi[a] \\ & 1 \end{pmatrix} g - g &= \sum_{\mu,\lambda \in \mathbb{F}_q} \left[\begin{pmatrix} \varpi^2 & [\mu] + \varpi[a] + \varpi[\lambda] \\ & 1 \end{pmatrix}, x^{\vec{r}-\vec{i'}} y^{\vec{i'}} \right] - g \\
&= \sum_{\mu,\lambda \in \mathbb{F}_q} \left[\begin{pmatrix} \varpi^2 & [\mu] + \varpi[a+\lambda] \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \frac{[a]+[\lambda]-[a+\lambda]}{\varpi} \\ & 1 \end{pmatrix} x^{\vec{r}-\vec{i'}} y^{\vec{i'}} \right] - g \\
&= \sum_{\mu,\lambda \in \mathbb{F}_q} \left[\begin{pmatrix} \varpi^2 & [\mu] + \varpi[a+\lambda] \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \varpi \cdot \frac{[a]+[\lambda]-[a+\lambda]}{\varpi^2} \\ & 1 \end{pmatrix} x^{\vec{r}-\vec{i'}} y^{\vec{i'}} \right] - g \\
&= \sum_{\mu,\lambda \in \mathbb{F}_q} \left[\begin{pmatrix} \varpi^2 & [\mu] + \varpi[a+\lambda] \\ & 1 \end{pmatrix}, x^{\vec{r}-\vec{i'}} y^{\vec{i'}} \right] - g \\
&= 0
\end{aligned}$$

For all $a, b, \mu \in \mathbb{F}_q$, let $t_{a,b,\mu}$ be the image of $[b] + \frac{[a]+[\mu]-[a+\mu]}{\varpi^2}$ in $\mathbb{F}_q$, then

$$\begin{aligned}
\begin{pmatrix} 1 & [a] + \varpi^2[b] \\ & 1 \end{pmatrix} g - g &= \sum_{\mu,\lambda \in \mathbb{F}_q} \left[\begin{pmatrix} \varpi^2 & [a] + [\mu] + \varpi^2[b] + \varpi[\lambda] \\ & 1 \end{pmatrix}, x^{\vec{r}-\vec{i'}} y^{\vec{i'}} \right] - g \\
&= \sum_{\mu,\lambda \in \mathbb{F}_q} \left[\begin{pmatrix} \varpi^2 & [\mu+a] + \varpi[\lambda] \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & [b] + \frac{[a]+[\mu]-[a+\mu]}{\varpi^2} \\ & 1 \end{pmatrix} x^{\vec{r}-\vec{i'}} y^{\vec{i'}} - x^{\vec{r}-\vec{i'}} y^{\vec{i'}} \right] \\
&= \sum_{\mu,\lambda \in \mathbb{F}_q} \left[\begin{pmatrix} \varpi^2 & [\mu+a] + \varpi[\lambda] \\ & 1 \end{pmatrix}, t_{a,b,\mu}^{p^{j_0}} x^{\vec{r}} \right] \\
&= T_+ \left(\sum_{\mu \in \mathbb{F}_q} \left[\begin{pmatrix} \varpi & [\mu] \\ & 1 \end{pmatrix}, t_{a,b,\mu-a}^{p^{j_0}} x^{\vec{r}} \right] \right) \in T_+ R'_1.
\end{aligned}$$

Since $\begin{pmatrix} 1 & \varpi[a] \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & [a] + \varpi^2[b] \\ & 1 \end{pmatrix}, a, b \in \mathbb{F}_q$ generate $U/\alpha^3 U \alpha^{-3}$, we see that $g \in (R_2(\sigma)/T_+ R'_1(\sigma))^U$ and $g \notin T_+ R_1(\sigma)$.

If $\dim_k(\sigma)=1$, $\vec{r}=\vec{0}$. We take $g=\sum_{\mu,\lambda\in\mathbb{F}_q}\left[\begin{pmatrix}\varpi^2 & [\mu]+\varpi[\lambda]\\ & 1\end{pmatrix},\lambda\right]$. Then $g\notin T_+R_1(\sigma)$ as $T_+R_1(\sigma)$ is spanned by $\sum_{\lambda\in\mathbb{F}_q}\left[\begin{pmatrix}\varpi^2 & [\mu]+\varpi[\lambda]\\ & 1\end{pmatrix},1\right]$ by (4.1). Then for any $a,b,c\in\mathbb{F}_q$,

$$\begin{aligned}
&\begin{pmatrix}1 & [a]+\varpi[b]+\varpi^2[c]\\ & 1\end{pmatrix}g-g\\
&=\sum_{\mu,\lambda\in\mathbb{F}_q}\left[\begin{pmatrix}\varpi^2 & [a+\mu]+\varpi[\lambda+b]\\ & 1\end{pmatrix},\begin{pmatrix}1 & [c]+\frac{[a]+[\mu]-[a+\mu]}{\varpi^2}+\frac{[b]+[\lambda]-[b+\lambda]}{\varpi}\\ & 1\end{pmatrix}\lambda\right]-g\\
&=\sum_{\mu,\lambda\in\mathbb{F}_q}\left[\begin{pmatrix}\varpi^2 & [a+\mu]+\varpi[\lambda+b]\\ & 1\end{pmatrix},\lambda-(\lambda+b)\right]\\
&=\sum_{\mu,\lambda\in\mathbb{F}_q}\left[\begin{pmatrix}\varpi^2 & [a+\mu]+\varpi[\lambda+b]\\ & 1\end{pmatrix},-b\right]\\
&=T_+\left(\sum_{\mu\in\mathbb{F}_q}\left[\begin{pmatrix}\varpi & [\mu]\\ & 1\end{pmatrix},-b\right]\right)\in T_+R_1'(\sigma)
\end{aligned}$$

since $T_-\left(\sum_{\mu\in\mathbb{F}_q}\left[\begin{pmatrix}\varpi & [\mu]\\ & 1\end{pmatrix},-b\right]\right)=\nu\left[\begin{pmatrix}1 & \\ & 1\end{pmatrix},\sum_{\mu\in\mathbb{F}_q}-b\right]=0$ by (4.2).

2) Assume F is unramified. Then $f>1$, $\varpi=p$. By the theory of Witt vectors, there exist polynomials $P_1,P_2\in\mathbb{Z}[x,y]$ such that for any $a,b\in\mathbb{F}_q$, $[a]+[b]\equiv[a+b]+p[P_1(a,b)]+p^2[P_2(a,b)]\mod p^3$. Since $P_1(a,b)=F(a^{1/p},b^{1/p})=F(a^{p^{f-1}},b^{p^{f-1}})$ where $F(x,y)=\frac{x^p+y^p-(x+y)^p}{p}$, we can assume P_1 is a polynomial of degree no more than $p^{f-1}(p-1)$ in each variable (or see Lemma 4.3).

If there exists $j_0\in\{0,\cdots,f-1\}$ such that $r_{j_0}+1\le p-1$ (i.e. $\vec{r}\neq(p-1,\cdots,p-1)$), we take

$$g=\sum_{\mu,\lambda\in\mathbb{F}_q}\left[\begin{pmatrix}p^2 & [\mu]+p[\lambda]\\ & 1\end{pmatrix},\lambda^{p^{j_0}(r_{j_0}+1)}x^{\vec{r}}\right].$$

We claim that $g\notin T_+R_1(\sigma)$. Otherwise, for each $\mu\in\mathbb{F}_q$, there exist $u_{\vec{i}}\in k$ for $\vec{0}\le\vec{i}\le\vec{r}$ such that

$$\sum_{\lambda\in\mathbb{F}_q}\left[\begin{pmatrix}p^2 & [\mu]+p[\lambda]\\ & 1\end{pmatrix},\lambda^{p^{j_0}(r_{j_0}+1)}x^{\vec{r}}\right]=\sum_{\lambda\in\mathbb{F}_q}\left[\begin{pmatrix}p^2 & [\mu]+p[\lambda]\\ & 1\end{pmatrix},(\sum_{0\le\vec{i}\le\vec{r}}u_{\vec{i}}(-\lambda)^{\vec{i}})x^{\vec{r}}\right].$$

Then $\lambda^{p^{j_0}(r_{j_0}+1)}=\sum_{\vec{i}\le\vec{r}}u_{\vec{i}}(-1)^{\vec{i}}\lambda^{\vec{i}}$ for every $\lambda\in\mathbb{F}_q$. This is impossible since the polynomial $X^{p^{j_0}(r_{j_0}+1)}-\sum_{0\le\vec{i}\le\vec{r}}u_{\vec{i}}(-1)^{\vec{i}}X^{\sum_{0\le j\le f-1}p^ji_j}\in k[X]$ is not zero and has degree no more than $q-2$ (by $f>1$ and $\vec{r}\neq(p-1,\cdots,p-1)$). For any $a,b,c\in\mathbb{F}_q$, we calculate that (using $x^{\vec{r}}\in\sigma^U$ and

$[a]+[b] \equiv [a+b]+p[P_1(a,b)] \mod p^2)$

$$
\begin{aligned}
&\begin{pmatrix} 1 & [a]+p[b]+p^2[c] \\ & 1 \end{pmatrix} g - g \\
&= \sum_{\mu,\lambda\in\mathbb{F}_q} \left[\begin{pmatrix} p^2 & [a]+[\mu]+p[\lambda]+p[b]+p^2[c] \\ & 1 \end{pmatrix}, \lambda^{p^{j_0}(r_{j_0}+1)} x^{\vec{r}} \right] - g \\
&= \sum_{\mu,\lambda\in\mathbb{F}_q} \left[\begin{pmatrix} p^2 & [a+\mu]+p[\lambda+b+P_1(a,\mu)] \\ & 1 \end{pmatrix}, \lambda^{p^{j_0}(r_{j_0}+1)} x^{\vec{r}} \right] - g \\
&= \sum_{\mu,\lambda\in\mathbb{F}_q} \left[\begin{pmatrix} p^2 & [\mu]+p[\lambda] \\ & 1 \end{pmatrix}, ((\lambda-b-P_1(a,\mu-a))^{p^{j_0}(r_{j_0}+1)} - \lambda^{p^{j_0}(r_{j_0}+1)}) x^{\vec{r}} \right], \qquad (4.3)
\end{aligned}
$$

Write $(\lambda-b-P_1(a,\mu-a))^{p^{j_0}(r_{j_0}+1)} - \lambda^{p^{j_0}(r_{j_0}+1)} = \sum_{0\le i\le r_{j_0}} g_i(\mu)(-\lambda)^{p^{j_0}i}$, where $g_i(\mu)$ are polynomials in μ (depending also on a,b).

First assume $p^{j_0}r_{j_0} \neq r = \sum_{j=0}^{f-1} r_j p^j$. For any $0\le i\le r_{j_0}$, let $\vec{i}_{j_0} = (i_1,\cdots,i_{f-1}) \in \mathbb{Z}^{f-1}$ such that $i_j = 0$ if $j\neq j_0$ and $i_{j_0} = i$. Then $\vec{i}_{j_0} < \vec{r}$ for any $i \le r_{j_0}$. Hence the last term in (4.3)

$$
\begin{aligned}
&\sum_{\mu,\lambda\in\mathbb{F}_q} \left[\begin{pmatrix} p^2 & [\mu]+p[\lambda] \\ & 1 \end{pmatrix}, (\sum_{0\le i\le r_{j_0}} g_i(\mu)(-\lambda)^{p^{j_0}i}) x^{\vec{r}} \right] \\
&\quad = \sum_{\mu\in\mathbb{F}_q} T_+ \left(\left[\begin{pmatrix} p & [\mu] \\ & 1 \end{pmatrix}, \sum_{0\le i\le r_{j_0}} g_i(\mu) x^{\vec{r}-\vec{i}_{j_0}} y^{\vec{i}_{j_0}} \right] \right)
\end{aligned}
$$

lies in T_+R_1' and we have found a required g.

Otherwise $r = p^{j'}r_{j'}$ for some j'. If $\vec{r}\neq 0$, we can choose in the beginning $j_0 \neq j'$ with $r_{j_0} = 0$ since $f \ge 2$ and $r_{j_0}+1 = 1 \le p-1$. Then $0 = p^{j_0}r_{j_0} \neq r$, we return to the previous case and we can find a required g. If $\vec{r} = 0$, we can let $j_0 = 0$, then $r_{j_0} = 0$. Then the last term in (4.3) is $\sum_{\mu,\lambda\in\mathbb{F}_q} \left[\begin{pmatrix} p^2 & [\mu]+p[\lambda] \\ & 1 \end{pmatrix}, g_0(\mu) \right] = T_+ \left(\sum_{\mu\in\mathbb{F}_q} \left[\begin{pmatrix} p & [\mu] \\ & 1 \end{pmatrix}, g_0(\mu) \right] \right)$. We have

$$
T_- \left(\sum_{\mu\in\mathbb{F}_q} \left[\begin{pmatrix} p & [\mu] \\ & 1 \end{pmatrix}, g_0(\mu) \right] \right) = \nu \left[\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \sum_{\mu\in\mathbb{F}_q} g_0(\mu) \right] = 0
$$

by Lemma 4.2 and $g_0(\mu)$ is a polynomial of μ of degree $(p-1)p^{f-1} < q-1$. Hence $ug-g \in T_+R_1'(\sigma)$ for any $u\in U$. We have found a required g.

(3) Now we remain the case when F is unramified over $\mathbb{Q}_p$, $f\ge 2$ and $\vec{r} = (p-1,\cdots,p-1)$. Let $\vec{i'} = (i'_0,\cdots,i'_{f-1})$ where $i'_j = 0$ if $j\neq 0$ and $i'_0 = 1$. Take

$$
g = \sum_{\mu,\lambda\in\mathbb{F}_q} \left[\begin{pmatrix} p^2 & [\mu]+p[\lambda] \\ & 1 \end{pmatrix}, x^{\vec{r}-\vec{i'}} y^{\vec{i'}} \right].
$$

Then $g \notin T_+R_1$ as $\vec{i'} \neq \vec{0}$. For any $a, b \in \mathbb{F}_q$, we calculate that (using $\begin{pmatrix} 1 & a \\ & 1 \end{pmatrix} x^{\vec{r}-\vec{i'}} y^{\vec{i'}} = a x^{\vec{r}} + x^{\vec{r}-\vec{i'}} y^{\vec{i'}}$)

$$
\begin{aligned}
&\begin{pmatrix} 1 & p[a]+p^2[b] \\ & 1 \end{pmatrix} g - g \\
&= \sum_{\mu,\lambda \in \mathbb{F}_q} \left[\begin{pmatrix} p^2 & [\mu]+p[a]+p[\lambda]+p^2[b] \\ & 1 \end{pmatrix}, x^{\vec{r}-\vec{i'}} y^{\vec{i'}} \right] - g \\
&= \sum_{\mu,\lambda \in \mathbb{F}_q} \left[\begin{pmatrix} p^2 & [\mu]+p[a+\lambda]+p^2[P_1(a,\lambda)]+p^2[b]+p^3\frac{[a]+[\lambda]-[a+\lambda]-p[P_1(a,\lambda)]}{p^2} \\ & 1 \end{pmatrix}, x^{\vec{r}-\vec{i'}} y^{\vec{i'}} \right] - g \\
&= \sum_{\mu,\lambda \in \mathbb{F}_q} \left[\begin{pmatrix} p^2 & [\mu]+p[a+\lambda] \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & [P_1(a,\lambda)]+[b] \\ & 1 \end{pmatrix} x^{\vec{r}-\vec{i'}} y^{\vec{i'}} \right] - g \\
&= \sum_{\mu,\lambda \in \mathbb{F}_q} \left[\begin{pmatrix} p^2 & [\mu]+p[\lambda] \\ & 1 \end{pmatrix}, (P_1(a,\lambda-a)+b) x^{\vec{r}} \right].
\end{aligned}
$$

$P_1(a, \lambda-a)+b$ is a polynomial of λ with degree no more than $p^{f-1}(p-1) < q-1$, the last term lies in T_+R_1' by the remark at the beginning ($\sum_{\lambda \in \mathbb{F}_q} \left[\begin{pmatrix} p^2 & [\mu]+\varpi[\lambda] \\ & 1 \end{pmatrix}, \lambda^{\vec{i}} x^{\vec{r}} \right]$ lies in $T_+R_1'(\sigma)$ if $\vec{i} < \vec{r}$).

For any $a \in \mathbb{F}_q$,

$$
\begin{aligned}
&\begin{pmatrix} 1 & [a] \\ & 1 \end{pmatrix} g - g \\
&= \sum_{\mu,\lambda \in \mathbb{F}_q} \left[\begin{pmatrix} p^2 & [a]+[\mu]+p[\lambda] \\ & 1 \end{pmatrix}, x^{\vec{r}-\vec{i'}} y^{\vec{i'}} \right] - g \\
&= \sum_{\mu,\lambda \in \mathbb{F}_q} \left[\begin{pmatrix} p^2 & [\mu+a]+p[\lambda+P_1(a,\mu)] \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & [P_2(a,\mu)]+[P_1(\lambda,P_1(a,\mu))] \\ & 1 \end{pmatrix} x^{\vec{r}-\vec{i'}} y^{\vec{i'}} - x^{\vec{r}-\vec{i'}} y^{\vec{i'}} \right] \\
&= \sum_{\mu,\lambda \in \mathbb{F}_q} \left[\begin{pmatrix} p^2 & [\mu+a]+p[\lambda+P_1(a,\mu)] \\ & 1 \end{pmatrix}, (P_2(a,\mu)+P_1(\lambda,P_1(a,\mu))) x^{\vec{r}} \right] \\
&= \sum_{\mu,\lambda \in \mathbb{F}_q} \left[\begin{pmatrix} p^2 & [\mu]+p[\lambda] \\ & 1 \end{pmatrix}, (P_2(a,\mu-a)+P_1(\lambda-P_1(a,\mu-a),P_1(a,\mu-a))) x^{\vec{r}} \right].
\end{aligned}
$$

$(P_2(a,\mu-a)+P_1(\lambda-P_1(a,\mu-a),P_1(a,\mu-a)))$ is a polynomial of λ of degree no more than $p^{f-1}(p-1) < q-1$. By the remark at the beginning, the last term lies in $T_+R_1'(\sigma)$.

Since $\begin{pmatrix} 1 & [a] \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & p[b]+p^2[c] \\ & 1 \end{pmatrix}, a, b, c \in \mathbb{F}_q$ generate $U/\alpha^3 U \alpha^{-3}$, $g \in (R_2(\sigma)/T_+R_1'(\sigma))^U$. Thus we have found a required g. □

Remark 4.4. *Those g in Lemma 4.1 have been found for many cases in* [4], [15], [14] *and* [9].

Corollary 4.5. *For any smooth irreducible representation σ of KZ, the universal supersingular representation of G $ind_{KZ}^G\sigma/T(ind_{KZ}^G\sigma)$ is not admissible if $F \neq \mathbb{Q}_p$.*

Proof Same as Corollary 2.21 in [16], using Proposition 4.5 in [8]. □

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